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# Basic Systems of Rational Norm-Curves

Dissertation

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By

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II

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Basic System of Ratio on the Cone

# I

## Introduction

1. The base, or $(n+2)$ point in a space of $n$ dimensions, derives its importance from the fact that it is the figure of the greatest number of independent points that is projectively equivalent to any other such set. The assumption of two ordered sets in two superposed or superposed spaces gives a collineation between these spaces.

The fundamental point-system $(1^n)$ in $n$ dimensions has $A^n_{n+2}$ elements. It is $n+1$ conditions

through a given point, and hence $K^n_5$ on $n+2$ points have

$$n^2+2n-3-(n-1)(n+2) = n-1$$

degrees of freedom. We shall call a system of $\infty^{n-1}$ $K^n$'s on $n+2$ points in $n$ dimensions a base system of rational curves, and individual curves of such a system, $K^n$'s. There is a unique $K^n$ on $n+3$ points, and hence through every point, except the base points, of the $n$ dimensional space $(S_n)$ under consideration passes one and only one $K^n$.

c. The following theorem is one of which use will be made throughout this paper. It is inserted here as a matter of convenience.

Basic $K^n$'s in $S_n$ having $p$ points contact with an $(n-1)$ way $\varphi^{n-1}$

$$(\alpha x)^m = 0,$$

of order $m$, touch at points of
($n$ $p$) way $\ldots$ of order

$$m(m+1)(m+2)\cdots(m+p-1),$$

which is the complete intersection
with a $\sigma_{\pi}$ $(n-1)$-way $\ldots$ of orders
$m+1, m+2, \ldots m+p-1$.

Proof; Take as base the $n+1$
points $\ldots$ the $\ldots$

[1]) Compare Humbert: Sur une Complique
Remarquable de Congrus, Journ. de Liouv.
Polytechnique, cah. 64 (1894) p. 123. Hum-
bert's requirement that basic $k$-fold
an $n$-ic surface at points of a curve of
order $2n^2$, and osculate in $6n^3$ points, is
obviously incorrect. Humbert equations
(6) should be

$$f(x, y, z, t) = 0$$
$$x^0 \frac{\partial f}{\partial x} + y^0 \frac{\partial f}{\partial y} + z^0 \frac{\partial f}{\partial z} + t^0 \frac{\partial f}{\partial t} = 0,$$

which is the form given previously.

unit point. An $R^n$ on these point and a point $\kappa$ is given parametrically by

$$x_i = \frac{\kappa_i}{1 - \kappa_i t} \qquad (1)$$

$t = \frac{1}{\kappa}$ gives the point of reference [illegible] gives the unit point, $t = 0$ gives the point

$$x_i = \kappa_i \prod_{j=1}^{n+1} (1 - \kappa_j t)$$

$$\frac{x_i}{\kappa_i} = \left[1 - S_{1,i} t + S_{2,i} t^2 - S_{3,i} t^3 + \cdots + (-)^n S_{n,i} t^n\right] \qquad (2)$$

where $S_{j,i}$ is the sum of products $j$ at a time of [illegible] $h_n^m(2)$ meets $(\alpha x)^m$ in the $m$ points given by

$$\left[(\alpha\kappa) - (\alpha\kappa S_1) t + (\alpha\kappa S_2) t^2 - \cdots\right]^m = 0 \qquad (3)$$

where

$$(\alpha\kappa) = \alpha_1\kappa_1 + \alpha_2\kappa_2 + \cdots$$

$$(\alpha\kappa S_1) = \alpha_1\kappa_1 S_{1,1} + \alpha_2\kappa_2 S_{1,2} + \cdots$$

Expanding (3) [illegible] ascending powers of $t$, we have

$$(\alpha\kappa)^m - m(\alpha\kappa)^{m-1}(\alpha\kappa S_1)\,t$$

$$+\Big\{\binom{m}{2}(\alpha\kappa)^{m-2}(\alpha\kappa S_1)^2 + m(\alpha\kappa)^{m-1}(\alpha\kappa S_2)\Big\}t^2$$

$$-\Big[\binom{m}{3}(\alpha\kappa)^{m-3}(\alpha\kappa S_1)^3 + 2\binom{m}{2}(\alpha\kappa)^{m-2}(\alpha\kappa S_1)(\alpha\kappa S_2)$$

$$+ m(\alpha\kappa)^{m-1}(\alpha\kappa S_3)\Big]t^3$$

$$+\Big[\binom{m}{4}(\alpha\kappa)^{m-4}(\alpha\kappa S_1)^4 + 3\binom{m}{3}(\alpha\kappa)^{m-3}(\alpha\kappa S_1)^2(\alpha\kappa S_2)$$

$$+\binom{m}{2}(\alpha\kappa)^{m-2}(\alpha\kappa S_2)^2 + 2\binom{m}{2}(\alpha\kappa)^{m-2}(\alpha\kappa S_1)(\alpha\kappa S_3)$$

$$+ m(\alpha\kappa)^{m-1}(\alpha\kappa S_4)\Big]t^4$$

$$- - - = 0 \qquad (4)$$

If $\kappa$ is a point of $(\alpha x)^m = 0$ $t$ is a factor of (4). If we desire the basic $K^n$ on $\kappa$ to touch the spread at $\kappa$, $t^2$ must be a factor of (4) and we must have

$$(\alpha\kappa)^m = 0$$

$$(\alpha\kappa)^{m-1}(\alpha\kappa S_1) = 0$$

If we desire $p$ points to coalesce, $t^p$ must be a factor of (4); the coefficients of $1, t, t^2 \ldots t^{p-1}$ in (4) must vanish. But these $p$ conditions in $(\alpha\kappa)$ only specify $p$ of order

$m, m-1, \ldots m-p+1$ [illegible]

The coefficients of (4) may be simplified by the identities,

$$
\begin{aligned}
(\alpha K S_1) &= (\alpha K)\,\sigma_1 - (\alpha K^2) \\
(\alpha K S_2) &= (\alpha K)\,\sigma_2 - (\alpha K^2)\,\sigma_1 + (\alpha K^3) \\
(\alpha K S_3) &= (\alpha K)\,\sigma_3 - (\alpha K^2)\,\sigma_2 + (\alpha K^3)\,\sigma_1 - (\alpha K^4),
\end{aligned}
\tag{5}
$$

&c.,

The forms (6) may be obtained immediately by substitution of the coordinates of $x$ as given by (1) in $(\alpha x)^n$, by permutating $p-1$ times as to $t$, and putting $t$ equal to 0 in the resulting expression. [illegible] spreads we have only to take the successive polar spreads of $\alpha$, and the modifications of the theorem for the case where $\alpha$ has coincident spreads are obvious. A specially interesting case arises when the basic $R^n$ has not $p$ point contact with $\alpha$ without lying entirely on it; we shall see several applications of this theorem in this connection.

We have from (6),

Basic $R^n$'s have $p$ point contact with an $S_{n-1}(\alpha x) = 0$, when

(1) Hamburger, [illegible]

$$(\alpha x) = 0$$
$$(\alpha x^2) = 0$$
$$(\alpha x^3) = 0 \qquad (7)$$

$$(\alpha x^{l}) = 0$$

## II

### Basic Norm Curves in Two Dimensions.

3. In a plane is a set of $\infty^2$ points, the basic system of $n$-ic curves is here the pencil of $n$-ics on the $n^2$ base points. These $n$-ics depend on the coefficients of any two of the planes. The $n^2$ points of this intersection are the points of contact of the $n^2$ tangents of the pencil that touch the line $(\alpha x) = 0$ in $n$ fixed points

of $\alpha$ that a basic conic with $\alpha$
are cut out by

$$(\alpha x^2) = 0.$$

Basic conics touch an $m$ in curve

$$(\alpha x)^m = 0 \qquad (1)$$

in the $m(m+1)$ points cut out by

$$(\alpha x)^{m-1}(\alpha x^2) = 0 \qquad (2)$$

If (1) is on a base point (2) touches (1) there. If (1) is on a point of the diagonal triangle of the base (2) passes through that point but does not in general touch (1) there. More generally, if (1) has an $i$-fold point at a base point (2) has an $i$-fold point there with the same tangents. If (1) has an $i$ fold point at a point of the diagonal triangle of the base (2) has also an $i$-fold point there, but not with the same tangents in general.

(2) reminder if the conditions for a node of a one sat field, that is, if

$$(px)^m \, {}'x_i = 0 \quad (i = 1, 2)$$

(where (7) passes through the nodes of $x$ and has to be deduced from the critical contact for each of the nodes.

If $m$ has a cusp at $x$ the system of lines

$$\frac{\partial w}{\partial x_1} \quad \frac{\partial w}{\partial x_2} \quad \frac{\partial w}{\partial x_3} = c$$

and

$$f^1 \frac{\partial w}{\partial x_1^2} + f^2 \frac{\partial^2 w}{\partial x_2^2} + f^3 \frac{\partial^2 w}{\partial x_3^2} + 2 f^{23} \frac{\partial^2 w}{\partial x_2 \partial x_3} + \dots$$

is a pencil space. This has a space with, if a pencil space, is the space of

$$f^{11} \frac{\partial^2 w}{\partial x_1^2} + f^{22} \frac{\partial^2 w}{\partial x_2^2} + f^{33} \frac{\partial^2 w}{\partial x_3^2} = 0, \quad (1)$$

the degree being three. This is the locus determined by the two systems

$$\left(\frac{\partial^2 \alpha}{\partial x_i^2}\right)\left(\frac{\partial^2 \alpha}{\partial x_j^2}\right) \qquad \frac{\partial^2 \alpha}{\partial x_i \partial x_j} \qquad (4)$$

(3) equated to zero is the condition that $\alpha = 0$ at $x$. Now the tangent to (3) at $x$ is, dropping the terms in $\frac{\partial \alpha}{\partial x_i}$:

$$\begin{aligned} &y_1\left(x_1'' \frac{\partial^2 \alpha}{\partial x_1^2} + x_2'' \frac{\partial^2 \alpha}{\partial x_1 \partial x_2} + y_3'' \frac{\partial^2 \alpha}{\partial x_1 \partial x_3}\right) \\ &+ y_2\left(x_1'' \frac{\partial^2 \alpha}{\partial x_1 \partial x_2} + x_2'' \frac{\partial^2 \alpha}{\partial x_2^2} + x_3'' \frac{\partial^2 \alpha}{\partial x_2 \partial x_3}\right) \\ &+ y_3\left(x_1'' \frac{\partial^2 \alpha}{\partial x_1 \partial x_3} + x_2'' \frac{\partial^2 \alpha}{\partial x_2 \partial x_3} + x_3'' \frac{\partial^2 \alpha}{\partial x_3^2}\right) = 0 \end{aligned} \qquad (5)$$

(5), on account of the condition (4), may be written

$$\left(y_1 \frac{\partial \alpha}{\partial x_1} + y_2 \frac{\partial \alpha}{\partial x_2} + y_3 \frac{\partial \alpha}{\partial x_3}\right)\left(x_1'' \frac{\partial \alpha}{\partial x_1} + x_2'' \frac{\partial \alpha}{\partial x_2} + x_3'' \frac{\partial \alpha}{\partial x_3}\right) = 0$$

Hence (4) passes through the conics of (3) and touches the tangents at the ranges. We have the

If $n$ points in a plane be related to the rays, from a point $x$ and $n$ rays, the number of conics having proper contact with the $2n$

$$N = n(n+1) \quad i.e. \quad 3n$$

We can also find the

of the curve of contact of bicircular conics with a pencil of m-ics. An m-ic of the pencil can touch the curve only at its points of contact with bicircular conics and at the base points of the pencil, since there is only one m-ic of the pencil on a point of the plane not a base point. If $k$ be the number of intersections of an m-ic with the curve

$$m^2 + m(m+1) - m(2m+1).$$

Hence follows the theorem

The curve of contact of a pencil of m-ics and a pencil of conics is a curve of order $2m+1$.[1]

If we have a pencil of m-ics

$$(xx)^m + \lambda(yy)^m = 0,$$

[1] This is a special case of a more general theorem, see Berzolari: Alg. Raum-Kurven, Encyk. d. math. Wiss. p. 633.

The locus of points common with the curves of this pencil are those common to curves of the pencil

$$(\alpha x)^{m-1}(\alpha x^2) + \lambda(\beta x)^{m-1}(\beta x^2) = 0$$

Eliminating $\lambda$ between these two equations, we have

$$\begin{vmatrix} (\alpha x)^m & , & (\beta x)^m \\ (\alpha x)^{m-1}(\alpha x^2) & , & (\beta x)^{m-1}(\beta x^2) \end{vmatrix} = 0, \qquad (6)$$

as the equation of the curve of contact. If $(\alpha x)^m$ have a node at a point $y$, $(\alpha x)^{m-1}(\alpha x^2)$ passes through $y$, and $y$ is a simple point of (6). Similarly, if $(\alpha x)^m$ have a $p$ fold point at $y$, $y$ is a $(p-1)$ fold point of (6) which passes through the $p$ points of the pencil at $y$ — i.e. $y$ is also on the four base points and the three pairs of base diagonals.

If Certain curves that pass through

correlation with the system of basic conics in a plane are of importance in the discussion of higher dimensional cases. We mention some of these here, though the discussion of the special cases of these curves that we shall meet later will be easier in the light of proofs that have not yet been brought forward.

A sextic curve $\Sigma$ is determined by the basic conics and a general conic $C$ in the following manner: through a point $P$ of $C$ draw the tangent to $C$. The locus of the point whose tangent meet the basic conics again is a rational sextic curve.

For [illegible]

at once two fixed [illegible] points being determined by the base conic on the pencil of tangency of tangents from the base point to C. Further, a basic conic, apart from the base points, can meet the curve only in the four points determined by its intersection with C, since there is one and only one basic conic through 5 points of the curve, and this must be its defining conic. We see, then, that no basic conic has $4 + 2 \cdot 4 = 12$ intersections with $\Sigma$, and hence it is of order 6, being met by a conic in twelve points. $\Sigma$ is in one-one correspondence with C and hence must be [illegible] C. If [illegible] touch C at the six points of contact of these

[illegible] the ten nodes of $\Sigma$ are at the base points. Since there is only one basic conic on a given point of the plane other than a base point, the only way in which additional nodes can arise is for the node on $C'$ of a [illegible] of $\Sigma$ and a basic conic $\gamma$ to be on $q$. If this happens [illegible] of $C$ and $q$, it will happen for the opposite common chord. We have then

Six nodes of $\Sigma$ [illegible] on the basic conics [illegible] of the pencil $\Sigma$

A unicursal curve in a plane has $\frac{1}{2}(n-1)(n-2)$ nodes, being [illegible] by a conic and a base [illegible]

$5 + 3 = 15$

constant. These conditions are to be connected for $\alpha$, the theorem of the modes. The condition of condition seems to be the existence of a perspective curve ($\Gamma^3$)

*) Staude, für einige gang der abzählenden rationalen Curven, Math. Ann. 21. Die rationale curven, die gerade in Bezug auf die acht in point, are in perspective position when they are in one one correspondence, and corresponding part and lines are incident. This requires, in general, that the two curves shall touch, or that the point curve shall have a cusp, at each common corresponding point. For let us say, that the line curve $A$ touch the point curve $B$ at a common corresponding point $C$ and $C$ coincides in a general tangent $t$

5. The curve in a plane determines a quadratic Cremona involution $x, y$ such that $x$ and $y$ are polars to each basis common. The pole-lines of $x$ with regard to the basis curve are on $y$. $x$ and $y$ are the points of contact of basis curve with the line $xy$. The basis points are the end a point of a line of $\Lambda$. This point approaches $a$ as the point of tangency of the tangent to $\Lambda$ approaches $a$. But a variable point on a tangent to a curve, as an ordinary point of the curve and in the neighborhood of the point of tangency cannot cross the tangent. Hence our statement. It is an easy inference from Steiner's theorem that a point curve and a line curve in perspective position must, in general, have $n^2 - n$ common corresponding points.

points of the transformation, and the three points of the diagonal triangle of the four points are singular points. A line is carried by the transformation into a conic on the singular points. A conic goes into a rational quartic curve on the singular points.

There are four pairs of corresponding points on any conic of the plane, namely, the eight points in which the conic meets its corresponding quartic. Assuming the basic conics as given, a pencil of conics $\Phi$ determines a series of pairs of points corresponding to each other on the same conic $\Phi$. This is subject to the condition that the pencil is determined by the four

conics which touch the pencil $\Phi$ at the point $x'y'$. The point of contact of conic $\Phi$ with the line $xy$ the locus of $x$ or $y$, is evidently on the base points of $\Phi$ since the condition $x'y'$ is possible if these base points. If $x$ is a base point of $\Phi$ the point $y$ determined by $x$ and the condition of $\Phi$ to the condition of $\Phi$ to $x'y'$. A conic $\Phi$ meets $S$ in $8+4=12$ point. Hence,

The locus of a point $y$, which corresponds to $x$ in the quadratic Cremona involution determined by the basic conics $\Phi$ and $S$ is a conic which passes through the points of the plane in a triple sense.

This conic is invariant under the quadratic involution, corresponding points on it being

interchanged. It follows that the invariant curves of the hexagon pass through the base points of the pencil.

These properties may be verified analytically. Let us assume as base the points $(1,1,1)\ (1,1,1)\ (1,1,1)\ (1,1,1)$ with diagonal points $(1,0,0)\ (0,1,0)\ (0,0,1)$. The Cremona involution determined by these conics is then simply

$$x_1' : x_2' : x_3' \qquad (7)$$

Suppose the pencil of conics $\Phi$ to be

$$\Phi = (\alpha x)^2 + \lambda(\beta x)^2 = 0 \qquad (8)$$

The points $x$ and $x'$ corresponding points lie on the same conic $\Phi$ which $\lambda$ must satisfy in addition, according to (7),

$$(\alpha/x)^2 + \lambda(\beta/x)^2 = 0, \qquad (9)$$

where $(\alpha/x) \equiv \alpha_1/x_1 + \alpha_2/x_2 + \alpha_3/x_3$.

Eliminating $\lambda$ between (8) and (9) we

Hence

$$S \equiv \begin{vmatrix} (\alpha x)^2 & (\beta x)^2 \\ (\alpha' x)^2 & (\beta' x)^2 \end{vmatrix} = 0, \qquad (10)$$

is the equation of our cubic. $(\alpha x)^2$ and $(\beta x)^2$ are conical pencils with nodes at the diagonal points of the base. The form of (10) gives us with regard to the cubic $S$:

1) $S$ is covariant under the collineation (7)

2) $S$ has nodes at the diagonal points of the base.

3) $S$ on the base point of both pencils of conics

The locus of point of contact of basic conics with the lines of a pencil is a curve of order $2+1+1=3$. This cubic is on the four base points and the three diagonal points. It is covariant

[illegible] corresponding to Chasles's [illegible] tions, points of contact in each [illegible] being in the tangent, every pair of the plane determine a conic of the pencil, so there are $\infty^2$ of them. They are all touching the three lines mentioned above. In order to construct one of this conic by a point it meets $\Delta$ in two [illegible] points apart from the vertex. This means that through any point $A$ of the plane there are four lines joining pairs of corresponding points on the conic $\Delta$. Hence

Since joining corresponding points of $\Delta$ touch a conic of class four.

The equation of this quartic may be written down at once by the Clebsch transformation [illegible]

two pencils of binary quadratics

$$(\alpha x)^2 + \lambda(\beta x)^2 = 0$$

$$(\gamma x)^2 + \lambda(\delta x)^2 = 0,$$

the apolarity condition of the Jacobian of the two pencils is

$$|\alpha\beta|\,|\gamma\delta|\,\{|\alpha\gamma|\,|\beta\delta| + |\alpha\delta|\,|\beta\gamma|\} = 0,$$

or

$$|\alpha\gamma|^2\,|\beta\delta|^2 - |\alpha\delta|^2\,|\beta\gamma|^2 = 0, \qquad (11)$$

since

$$|\alpha\beta|\,|\gamma\delta| \equiv |\alpha\gamma|\,|\beta\delta| - |\alpha\delta|\,|\beta\gamma|$$

the equation of our quartic is, from (11),

$$|\alpha\gamma\xi|^2\,|\beta\delta\xi|^2 - |\alpha\delta\xi|^2\,|\beta\gamma\xi|^2 = 0, \qquad (12)$$

this being the equation of the quartic determined by the two pencils of conics

$$(\alpha x)^2 + \lambda(\beta x)^2 = 0$$

$$(\gamma x)^2 + \lambda(\delta x)^2 = 0.$$

If $(\gamma x)^2 + \lambda(\delta x)^2$ be the pencil of conics on the base $(1\ 1\ 1)$ we have

$$\gamma_{ij} - \delta_{ij} = \gamma_{ii} - \delta_{ii} = 0,$$

and (12) becomes

$$\Sigma(\alpha_{22}\beta_{33} - \alpha_{33}\beta_{22})\xi_1^4$$
$$+ 2\Sigma(\alpha_{11}\beta_{23} - \alpha_{23}\beta_{11})\xi_1\xi_2^2(\xi_2 - \xi_3) \qquad (13)$$
$$+ \Sigma[\alpha_{11}(\beta_{33} - \beta_{22}) - \beta_{11}(\alpha_{33} - \alpha_{22})]\xi_2^2\xi_3^2$$
$$+ 2\Sigma[\alpha_{23}(\beta_{33} - \beta_{22}) - \beta_{23}(\alpha_{33} - \alpha_{22}) + 2(\alpha_{12}\beta_{13} - \beta_{12}\alpha_{13})]\xi_1^2\xi_2\xi_3$$
$$= 0.$$

The terms may [illegible] by cyclic permutation of the [illegible]

The equation (13) shows that the quartic touches the six lines $(1, \pm 1, 0)$ $(1, 0, \pm 1)$ $(0, 1, \pm 1)$ the six lines joining the base point. By symmetry we infer that it also touches the six lines joining the base points of $\Phi$.

It will be observed that [illegible] are determined by the [illegible] and the points $\Phi$, $\Phi'$, whose equations are [illegible] and the [illegible] the locus of pairs of corresponding points $x'y'$ which lie

on basic conics. The equation of $S'$, may be obtained as was found that of $S$. It is, in determinant form,

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \end{vmatrix} = 0 \qquad (14)$$

where

$$y_i = \frac{\partial}{\partial a_i} |a\alpha\beta|(\alpha x)(\beta x), \qquad (15)$$

(13') being the Pentasse in which is determined by the conics $\Phi$. The quartic (13) is symmetrically related to both pencils of conics and to both sextics. Its lines meet the two sextics in hermanic pairs

## III

### Basic Norm-Curves in three Dimensions.

6. The case in three dimensions is a set of five points. There is a system of $\infty^2$ norm curves ($R^3$'s) on the five points, since an $R^3$ has twelve coördinates, and it is ten conditions on a curve in space to pass through five given points. In addition we have a linear system of $\infty^9$ quadric surfaces on the five base points, which we shall call basic quadrics. In our discussion of the subject we shall make use of the following known theorems.

a) Basic quadrics cut any plane $\pi$ in conics of a pencil. To a definite conic $C_2$ in $\pi$ [1]

b) Basic $R^3$'s cut a net of the

[1] Compare Veriu, Nouvelles considérations sur la théorie des

point which are the vertices of a self polar triangle in $C_2$.

The use of these points is very important in the determination of the apparent deficiency of curves determined by basis $K^{(3)}$'s on a plane. We shall use the symbol $\delta^{(3)}$ in referring to curves of a set of points. In general we shall use the symbol $\delta_2^{(3)}$ in referring to a set of $\infty^2$ points, i.e. a curve $\phi^{(3)}$ in a double manifold, or to any flat under consideration.

c) Basis $K^{(3)}$ tangent to a bunch at point of $C_2$.[1]

d) Six bunch $K^{(3)}$'s osculate at the point of osculation being on $C_2$.[2]

1) Reye, Geom. der Lage, part II p. 223 ff.

2) Sturm, Crelle's Journal, vol. 79, p 99. There are many points of contact between the pairs of these curves and the pairs

e) Four basic $R^{(1)}$'s tangent to a cubic
a system in points of a rational
elliptic curve with nodes at the four
points of the Desargues configuration
(configuration 15) determined as
by the five base points [1]

f) Basic elliptic quartic ($E^{(4)}$) meeting
a in set of four points orthic to $C_\alpha$. [2]

By a set of four points orthic to
$C_\alpha$ we mean four points such that
any conic on them taken in points
is polar to $C_\alpha$ taken in lines. Dually
finally we have sets of four lines orthic
to $C_\alpha$.

---

Of the ... of either cubic on the one
hand, and the pencil section of the sup-
per on the other, but the point of view
is somewhat different.

[1] See ...

[2] Also ... Configuration of ...

... may be defined as the ... will regard it when the ... is determined on a ... by the ... is self-polar. We name the points of the base 1,2,3,4,5, the ten lines joining two of these five points 12, etc., and the ten planes joining three of them 123, etc., and name the points, lines of the configuration $\beta$ or $\alpha$ after the lines and planes of the base by which they are determined.

7. Consider the whole set of basis $R^3$'s as projected from a base point, say 5. Each gives us a pencil of quadric cones with vertex at 5 and on the four lines 15, 25, 35, 45. The ... of these cones thru a ... infinity of ... $R^3$'s, and $R^3$ ...

every point of the cone (a conic) $\Gamma^2$ through a point $a$ of one of these cones [illegible] on the cone, since it has seven points in common with it, [illegible] two at 5, and one at $a$.

This pencil of quadric cones determines on $\kappa$ a pencil of conics $\gamma$ 15, 21, 31, 45, all of which are apolar to $C_a$ by theorem a). 15, 17, 35, 45 are an orthic 4-point as to $C_a$. A defining characteristic of such a set of points is that the pole of the line joining any two of the four points is on the opposite base of the complete four-point of conics of the pencil, being apolar to $C_a$, must contain a single infinity of these four-points apolar to $C_a$ [illegible] not a base point of the pencil of

determine uniquely a basic $K^3$, and since one point of a conic of the pencil is sufficient to determine uniquely its three-point apolar to $C_\alpha$ we have the theorem,

*Three points $L^{(1)}$ on $\alpha$ are three-point in-scribed in conics of the pencil 15, 25, 35, 45, and apolar to $C_\alpha$, and conversely.*

In the configuration 13 there are five 6-three power-points, namely,

| | | | |
|---|---|---|---|
| 15 | 25 | 35 | 45 |
| 14 | 24 | 34 | 54 |
| 13 | 23 | 43 | 53 |
| 12 | 32 | 42 | 52 |
| 21 | 31 | 41 | 51 |

Here any [illegible] exceed the set 5, 4, 3, 2, 1 respectively. Since $aC_\alpha$ and $\beta$ have one point and only one in common, — the [illegible]

[illegible] Conic, then, must in only three variable points, and these set of three points are those for $C_5^{(1)}$. [illegible] the theorem.

The three point of a set $C_5^{(1)}$ [illegible] a conic with any [illegible] four [illegible] of 15.

A set of point $C_5^{(1)}$ may be constructed when one of the three points is given by drawing the conic from 15, 25, 35, 45 and the given point. The other two point are the intersections with this conic of the polar line of the given point as to $C_2$. This gives us at once theorem (1) above and the following:

Basic $N^{31}$ osculates at the six point of contact with $C_2$ of conic on 11, 21, 31, 41.

Tangent of six [illegible]

of B touch $C_x$ at the same six points.

The cubic of theorem c) is the locus of the points in which the tangents to $C_x$ at a point of a conic of a given cubic pencil of B meet the latter conic again.

8. Let us examine this matter a little more closely. It was in part II that a cubic generated in this way has made on the base points of the generating pencil and to the $C_x$ at the six points of contact of tangent conics of that pencil. Since the pair of each line of an order four point is on the opposite line, each a line taken with its opposite line is a degenerate conic of the pencil, the six tangents at the points of the base are each counted four times, once for each [illegible]

4).

Considering 15, 25, 35, 45, as a base in $\Sigma$, the equation of this cubic may be found by demanding that the polar line of a point $x$ as to the basic conic on $x$. Choosing as reference points 15, 25, 35 with 45 as $(1,1,1)$, a basic conic may be written

$$m_1 x_2 x_3 + m_2 x_3 x_1 + m_3 x_1 x_2 = 0 \quad (1)$$

with the condition

$$m_1 + m_2 + m_3 = 0 \quad (2)$$

The net of 6 of the form

$$a_1 \xi_1^2 + a_2 \xi_2^2 + a_3 \xi_3^2 + 2\lambda(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) = 0, \quad (3)$$

the apolarity condition of (1) and (3) being found to be

$$\lambda(m_1 + m_2 + m_3) = 0,$$

which vanishes by (2). The point equation of (3) is

$$(cx)^2 \equiv (a_2a_3 - \lambda^2)x_1^2 + (a_3a_1 - \lambda)x_2^2 + (a_1a_2 - \lambda^2)x_3^2 \quad (4)$$
$$+ 2(\lambda^2 - \lambda a_1)x_2x_3 + 2(\lambda^2 - \lambda a_2)x_3x_1 + 2(\lambda^2 - \lambda a_3)x_1x_2 = c$$

The basic conic $c$ is projected into

$$y_i = \frac{x_i}{1 - x_i t} \qquad (i = 1, 2, 3) \qquad (5)$$

The polar line of $x$ as to $(cx)^2 = c$ is

$$y_1 \frac{\partial c}{\partial x_1} + y_2 \frac{\partial c}{\partial x_2} + y_3 \frac{\partial c}{\partial x_3} = 0 \qquad (6)$$

Substituting (5) in (6), we have the transformation in $t$

$$\frac{\partial c}{\partial x_1}\frac{x_1}{1 - x_1 t} + \frac{\partial c}{\partial x_2}\frac{x_2}{1 - x_2 t} + \frac{\partial c}{\partial x_3}\frac{x_3}{1 - x_3 t} = 0$$

or

$$\frac{\partial c}{\partial x_1} x_1(1 - x_2 t)(1 - x_3 t) + \frac{\partial c}{\partial x_2} x_2(1 - x_3 t)(1 - x_1 t)$$
$$+ \frac{\partial c}{\partial x_3} x_3(1 - x_1 t)(1 - x_2 t) = 0$$

which, arranged in powers of $t$, is

$$x_1x_2x_3 \sum \frac{\partial c}{\partial x_1} t^2 - \sum x_1x_2\left(\frac{\partial c}{\partial x_1} + \frac{\partial c}{\partial x_2}\right) t + 2c = 0, \quad (7)$$

since $x_1 \frac{\partial c}{\partial x_1} + x_2 \frac{\partial c}{\partial x_2} + x_3 \frac{\partial c}{\partial x_3} \equiv 2c$.

The condition that (6) touch (5) is the discriminant of (7). The equation of the [illegible]

$$8x_1x_2x_3 \frac{\partial c}{\partial x_1} c - \left\{\sum x_1x_2\left(\frac{\partial c}{\partial x_1} + \frac{\partial c}{\partial x_2}\right)\right\}^2 = 0$$

This is of the form

$\delta x_1 x_2 x_3 L C^0 - f_3^2$ , (8)

where $f_3$ is a cubic ... $L$ is the polar line of (1,1,1) or 45° ... to $C_\alpha$, a conjugate line. (8) obviously touches $C_\alpha$ at its intersections with $f_3$. Hence $f_3$ is a cubic on the six points of osculation of ... with $C^0$'s polar ...

The four points 1°, 2°, 3°, 4°, and the conic $C_\alpha$ are sufficient to determine the configuration $B$ on $\alpha$. The points of $B$ are the four base points and the poles with to $C_\alpha$ of the six lines joining them. The lines are these six lines and the polar lines of the base points. The coördinates of the points of $B$ are

1° $1, 0, 0$

2° $0, 1, 0$

3° $0, 0, 1$

4° —

14 $x_1, \lambda, \lambda$ 23 $0, x_2, \lambda$. $(x_2 \lambda)$

24 $\lambda, x_2, \lambda$ 31 $(x_3, \lambda), 0, x_3 \lambda$

34 $\lambda, \lambda, x_3$ 12 $x_1, \lambda, -(x_1 \lambda)$

If you do this table the number of the points in the various groups

(8) Show that the cubic $f_3$ meets the lines $x_1, x_2, x_3, \lambda$, at points of the $x\lambda$. This gives the theorem,

The six points of contact of basic pairs with $f_3$, on the six lines of reciprocation of basic $f_3^2$ with $\lambda$, and the six points $12, 23, 31, 12, 23, 31$, are on a conic locus $\varphi$

$f_3$ is in full,

$$f_{3/2} = a_3(a_2 \lambda) x_1^2 x_2 + a_2(a_3 \lambda) x_1 x_3^2 + a_1(a_3 \lambda) x_2^2 x_3 + a_3(a_1 \lambda) x_2 x_1^2 + a_2(a_1 \lambda) x_3 x_1^2 + a_1(a_2 \lambda) x_3 x_2^2 + \left[6\lambda^2 - 2\lambda(a_1 + a_2 + a_3)\right] x_1 x_2 x_3 = 0 \qquad (10)$$

(10) is, in terms of the cubic

$$(cx)(cx^2) = 0,$$

which by equations (6), page 6, cuts out the contacts of basic conics with $C_\alpha$,

$$f_{-1} = (x_1 + x_2 + x_3)(\quad) - (\quad)(\quad) \qquad (11)$$

The above theorem is immediately evident from (10) and (11) by actual substitution of the coördinates of the six points as given in the table (1). These six points are the vertices of the 4 lines of 25, viz. 135, 124, 123. This is the polar reciprocal as to $C_\alpha$ of the conic 4-point 4, and is a conic 4 line as to $C_\alpha$.

There can be but one cubic with nodes at the ten points of a configuration 15, two would have forty points of intersection. Although one system of equations is unsymmetrical, the sextic is symmetrical with regard to the

The six lines of any orthic 4-point of $B$ and the six lines of contact with $C_x$ of conics on the four points, touch a conic of class three.

A metrically special, but projectively general, rational cubic with nodes at the ten point of a configuration $B$ may be easily constructed by choosing 15 25 as the circular points in the plane; 35, 45 are then isotropic point, in the director circle of $C_x$, and the pencil of conics on 15, 25, 35, 45, is then a pencil of circles on two points.[1]

9. From equations (6'), page 6, we have, the four reference points and the unit point being taken as base, —

Basis $N^3$ touches the $m$-face

$$(\alpha x)^m = 0$$

[1] Siddon, Conic Sections, p. 341, ex. 2.

in general if the equation of order $(m+n)$ has one of $(\alpha x)^m$ as the equation

$$(\alpha x)^m \, \phi(\alpha x) = 0 \qquad (12)$$

and also has $(\alpha x)^n$ as the $m(m+1)(m+2)$ point of intersection of the two curves

$$(\alpha x)^m = 0$$
$$(\alpha x)^m \, \{(\alpha x)^2\} = 0 \qquad (13)$$
$$(m-1)(\alpha x)^{m-2}\{(\alpha x)^2\}^2 + 2(\alpha x)^{m-1}(\alpha x)^1 = 0$$

(14) is on all nodes of $(\alpha x)^m$.

Hence,

The locus of points of contact of curves of $m^{th}$ order which can in general pass through $\frac{m(m+3)}{2}$ nodes of the curve is the curve itself, because this curve factors twice out of the locus of Contact & multiple factor of the curve is a factor three times and of the locus of Contact

the following is easily verified:

If $(\alpha\alpha)^r$ passes $k$-times through $p$ and $(\beta\beta)^s$ $l$-times, $(\alpha\alpha)^r(\beta\beta)^s$ does so $(k+l)$-times, and the tangent of contact has a point at the point $p$.

10. The known theorem that there is one and only one basic $R^3$ which meets a given line $\pi$ of space twice, is an immediate consequence of theorem 9. If we choose any plane $\alpha+\lambda\beta$ of the pencil of planes containing $\pi$, the $R^3$ on the base points and the pole of $\pi$ as to the conic $C_{\alpha+\lambda\beta}$ in this plane must meet $\pi$ twice, since the set of points $(1,3)$ determined by this pole on $\alpha+\lambda\beta$ is apolar to $C_{\alpha+\lambda\beta}$. The locus of this pole, as $\alpha+\lambda\beta$ turns around $\pi$, may or shown independently to be a cubic $R^2$; we shall do this later. We shall call the locus $R^2$ the

11. In the plane $\alpha + \lambda\beta$ there is a conic $\pi$ the locus of $C_{\alpha+\lambda\beta}$ passing through the five base points and touching $\pi$. If a plane is on a base point the corresponding $C$ on that plane is made up of a complete four-line, taken with the tangent at the base point, and the cubic $C_\alpha$ in the case the line $\xi\xi'$ through the base point which are further in the Cremona line involution determined by the pairs of cubics on the four lines. It follows that the planes joining $\pi$ to each of the base points are triple tangent planes of the cubic surface the degenerate conics $C_{\alpha+\lambda\beta}$ of these planes form the locus of the surface

$\alpha_1 \alpha_2 \alpha_3$.

The conic $C_\alpha$ on a plane $\pi$ is composed of the points $x$ of the plane such that

$$(\alpha x^2) \quad 0^{(1)}$$

The conic $C_{\alpha+\lambda\beta}$ passes through the points

$$(\alpha x) + \lambda(\beta x) = \qquad (14)$$

in common with the polars by the corresponding quadric

$$(\alpha x^2) + \lambda(\beta x^2) = 0 \qquad (15)$$

Eliminating $\lambda$ from (14) and (15) we have as the locus of $C_{\alpha+\lambda\beta}$ the cubic curve

$$\begin{vmatrix} (\alpha x^2), & (\beta x^2) \\ (\alpha x), & (\beta x) \end{vmatrix} = \sum \pi_{ij} x_i x_j (\alpha_i - \alpha_j) = 0, \qquad (16)$$

the Pascal line of $6$ points determined by the line

$$\pi_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$$

Similarly $(\alpha\alpha_i\alpha_j)$ gives the tangent of $C_\alpha$ at $\alpha_i$ also be inferred from the fact that the line

and the base.

The combinations $x_i(x - x_j)$ in (16) are [illegible]. The [illegible] $R^3$ are $x = \omega$

$$\gamma \quad 1 \quad x_i t$$

where

$$\frac{dx}{dt} \quad (x_i t)$$

$t = 0$ giving the point $x$. It follows that the basic $R^3$ on $x$ passes thro $x$ with the tangent

$$p_{ij} = x_i x_j (x_i - x_j) \qquad (11)$$

(10). The [illegible] the [illegible] the tangent [illegible] $\pi$. [illegible]

[illegible] $(10)$ is the [illegible] $x$ and [illegible] tangent [illegible] $x$ [illegible] $x$ [illegible]

[illegible]

Any degenerate line of order two.

Any base line $l_2$ taken with a
certain one $S_2$, and the pencil order
$n$ about $S_2$, is a degenerate locus
$K^2$. A tangent to $K^2$ meets $K^2$
except at the points $S_2$, $n$ times, & is
degenerating in these ways. The center
of the $n$-tuple point $(n)$ of $K^2$ is
moreover, then, on the cubic curve of
contact of tangents of this second
order cone of the pencil on the
point in which it meets $S_2$.

13. If a point $x$ of a locus $K^2$
moves along a line $p$, the tangent to
the corresponding $K^2$ at $x$ generates a
pencil of the $n^{th}$ order corresponding to
the line $p$.

For the pencil of tangent planes
generated by any plane $p$ of space
are the lines in the base plane of

in the two tangent to $\Gamma^2$ at the points where the line meets the curve $\Gamma_2$. It is clear that every such surface is determined by the line $\rho$ by the equation (16).

The cubic surface (16) is general and contains [illegible] lines. If the curve $\Gamma_2$ has a double point $O$ then the line $\rho$ and the two lines $\sigma_1, \sigma_2$ of the cone (16, $\rho$) say. All tangents to the curve $\Gamma^2$ at points $x$ of $\rho$ meet $\pi$. $\pi$ must then be contained in the locus surface $(\Gamma^2, \rho)$. Consider a section of $(\Gamma^2, \rho)$ by any plane $\pi$ of $\pi$. The line $\pi$ and the tangent to the curve $\Gamma^2$ on the point where $\rho$ meets $\pi$ are the only lines that can be included in this section, since the line $\pi$ must meet the

Again, a ruled cubic surface cannot contain more than two lines meeting all generators, and one of these must be the double line. If three lines must all generators of a ruled surface the surface is a quadric. We have then the theorem,

16. Since $\pi$ cuts the cubic surface (16) is the double line of any ruled cubic surface [16] determined by any three of the surface $S_{\pi}$.

14. If $\pi$ is a tangent to a $R^3$ the cubic surface (16) has a node at the point of tangency, and (16) is the locus of the nodal cones the line $\pi$ and the lines to the five base points. Since an $R^3$ is projected from any point on it by a quadric cone we have

Theorem.

The six lines through the points of a twisted cubic form two generators of a quadric cone.

15. We see from (17) that the tangents to all twisted cubics depend on the variation of four homogeneous parameters, that is, they lie in a complex. We proceed to determine its order. Suppose the $p_{ij}$ to be homogeneous coördinates of a point in five-dimensional space. The lines of space are represented by the points of the quadric ground,

$$Q = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$$

(17), considered as a locus of points in five dimensions, is evidently a three-way ground form entirely on $Q$. The four planes $x_i = 0$ and the six $x_i = x_j$ are the tangent

joining the two points $y_i$ to the nodes of these planes on an edge of the tetrahedron of reference. All the $p_5$ considered as cubic surfaces contain the five base points and the ten points in which a line meets the opposite plane. Hence,

$$c_{ij} p_i p_j = 0 \qquad (18)$$

is a cubic surface on these fifteen points. Now the order of a thing way spread in five dimensions is the number of points in which it is met by an arbitrary plane, which may be represented analytically by three equations of the form (18). Three of the cubic surfaces (13) meet in

$$27 - 15 = 12$$

base points. Hence we have a pencil

spread of order twelve, and the complex must be of order six. We have then

(I) the locus of a tangent to $k^{(3)}$ and $k^{(4)}$ is a cubic complex

(II) is a system of conics on fifteen points, but is not a five fold system as we should expect. The fifteen points, since the property that any two are conjugate in four systems of them must be on the pfaffian, is just enough to fix it, without building up the equation of the cubic surface on quintuples of the points. (III) is merely the locus of points of tangency of rays $k^{(3)}$ with lines of the complex of tangents that lie in a given linear complex. (IV) is obviously a special case, the Steiner, but it

linear [illegible] through the [illegible] that
[illegible] in [illegible] $\pi$.

16. The equation of the [illegible]
of tangents to the basic $K^3$ may be
obtained by elimination of the $x_i$
from (17), but it is more easily
found directly. The form of equa-
tion (15) shows that $\pi$ is an axis of
the pencil of planes $\alpha+\lambda\beta$ in such
a $\Gamma^3_{\alpha+\lambda\beta}$ in pairs of an involution.
This fact is also contained in the-
orem 6), which shows that points
of $C_{\alpha+\lambda\beta}$ on $\pi$ are apolar to the
points of intersection with $\pi$ of
its biscant basic $K^3$; hence the
[illegible] $K^3$ [illegible] the [illegible]
[illegible] of this involution. If the
[illegible] basic $K^3$ [illegible] $\pi$ the
two double points of the involution
on $\pi$ [illegible] and the [illegible]

of the complex of tangents to conic
$K^2$ may be obtained by expressing
the condition that the ... vector
$\alpha + \lambda\beta$ is tangent. Represent the points
of plane conic $K$ by $\alpha x^2$, the
tangents by $\alpha u^2$. Then the
relation of point and tangent
$(\alpha + \lambda\beta)$ ... $K$ is ... ,

$$(\alpha x^2) + \lambda(\beta x^2) = 0.$$

The double points of this involution
are given by the Jacobians of the
quadratics in $\mu$

$$(\alpha x^2)(\beta x^2) = 0,$$

where $x + \mu x'$ is substituted for $x$.
The discriminant of the Jacobian
is

$$\left[(\alpha a^2)(\beta b^2) - (\alpha b^2)(\beta a^2)\right]^2$$
$$- 4\left[(\alpha a^2)(\beta ab) - (\beta a^2)(\alpha ab)\right]\left[(\alpha - )(\alpha b^2)(\beta ... )(\beta ... )\right] = 0$$

... writing in this

$$\pi_{ij} = \alpha_i \beta_j - \alpha_j \beta_i = p_{ik} \quad \text{for} \quad ik - jk$$

we have as the equation of the complex

$$\sum_{12}^{k} p''_{12} p''_{34} - \lambda \sum_{12} p'_{12} p''_{34} p_{12} p_{43} + \lambda^2 \sum_{12} p'_{12} p''_{34} (p_{14} p_{23} + p_{13} p_{42}) = 0$$

17. There being a $\infty^{6}$ of $K^{1}$'s, of a single condition, in space $S_3$, a single condition is imposed on a curve $K^{2}$ if it is to lie on a surface of $S_3$; it is important to this connection, [illegible] that a $K^{2}$ can pass only in pencils $K^{2}$'s and in nets of $K^{2}$'s, i.e., in base lines.

Sturm's proof of the Since it applies [illegible] to higher cases. If $x$ is a point of intersection of the two surfaces the [illegible] $K^{2}$ on $x$ cannot lie in both surfaces, but the $K^{2}$ is completely determined by $x$, and since the surfaces being of [illegible]

there ... of the ... surface through the point x.

As an illustration of the ... we prove that there are ... a point within ... manner. ... determined by π contains each of the ten base lines singly; the basic $K^3$ of which it is a point is ... the ... of the ... the ... the ... the ...

a curve of order $n^2$, $n$ being the order of each surface. In this intersection are the ten base lines and $n$ basic $K^3$'s. This gives the equation,

$$n^2 - 10 = 3n \quad \text{or} \quad n = 5.$$

18. We have, then, that basic $K^3$, which must a given one $\pi$ be a quintic surface. The intersection of $K^3$ by $\pi$ is [illegible] a conic curve of this surface. The surface contains the ten base lines, and has triple points at the five base points. It may be pointed out in passing that the surface is rational, and is mapped from a plane by [illegible] the curves on the ten points of it to the system of five lines and the eleventh point $K$.[1] The point $K$ is [illegible]

[1] Compare Clebsch, Math. Ann. 1.

into the line $\pi$. The pairs lines in the plane map into the pairs $C_1$ and the two points of intersection of the lines with the conic. The lines tangent from $K$ to the conic correspond the pairs lines which map into the conic $K'$ tangent to $\pi$, points of these two lines which map into the same point of the surface being determined by tangents to the conic.

This quintic surface is met by a general plane in a quintic curve with three nodes. If is met by a plane $\alpha + \lambda\beta$ on $\pi$ in the line $\pi$ and a quintic curve with a node at the point $H_1$ of $\pi = 0$ ($\alpha, \beta$). If is thus proved by this that the points of the corresponding points on the quintic (point lying on the cone basis $R^3$) are polar lines of po-

of $\pi$ to $C_{x+\lambda 3}$ and hence passes through $M$. Since one of its projectile points is any point of it by a general law, so

This quartic is the locus of points on $x+\lambda 3$ from which the five base points and the point $M$ are projected by six lines of a hyperboloid. It is the section by $x+\lambda 3$ of the Weddle quartic surface with nodes at the six points $1, 2, 3, 4, 5, M$, — the section of a Weddle quartic through a node. It meets $\pi$ in the two points of $C_{x+\lambda 3}$ and in the two points of intersection with $\pi$ of its six-point conic $K^2$, these four points forming a harmonic pencil.

Any point $P$ of space determines with regard to our $K^2$ a pencil $P'$

such that $P$ and $P'$ are apolar to all quadrics on the $K^3$. $PP'$ is a bisecant line of the $K^3$ and $P$, $P'$ are apolar to the pairs of intersections of the $R^2$ with $P_1$. Since $\pi$ may be considered as any of the $\infty^3$ lines of the $\Gamma^3$ on $(1, 2, 4, 8)$, we see from the above that the transformation $PP'$ sets up of the $\Gamma^3$ on itself a transformation of Weddle surface unaltered.[1])

The six points of tangency of tangents from $M$ to the quartic in $\alpha + \lambda\beta$ are on $C^6_{\alpha+\lambda\beta}$. There are six tangents through $M$ to the complex curve of (19) on $\alpha + \lambda\beta$. The complex curve of these tangents on any plane $\pi$ is the polar reciprocal of the Steiner figure of the six points on the conic $C^2_\pi$.

[1]) Schouten, Mess. of Math., 51, 1921, p. 21.

The complex cone of rays through any point $M$ is the cone joining $M$ to the twisted cubic determined by the five base points and $M$. The points of contact of rays from $M$ are points of contact with the surface.

19. We have made the statement that the locus of the pole of a line $\pi = 0$ w.r.t. $C_{\alpha+\lambda\beta}$ in a plane $\alpha+\lambda\beta$ containing $\pi$, as $\alpha+\lambda\beta$ turns about $\pi$, is a curve $\Gamma^3$ meeting $\pi$ twice. The order of the locus may be found by considering its points of intersection with a plane $\alpha+\lambda\beta$. These points are the pole of $\pi$ w.r.t. $C_{\alpha+\lambda\beta}$, and the points of the conic in which $\alpha+\lambda\beta$ cuts $\Phi$. Since a point of the locus is to be on $\pi$, $\pi$ must touch a conic $C_{\alpha+\lambda\beta}$. But this happens twice, since these

[illegible] that [illegible] [illegible] of [illegible] [illegible] the [illegible] the [illegible] [illegible] $R^2$ [illegible] a [illegible]. [illegible] since the pole of [illegible] to the degenerate [illegible] on the plane joining it with a [illegible] [illegible] the [illegible] of [illegible] [illegible] the [illegible] points [illegible].

(3). We have seen that [illegible] $R^2$'s touch an $n$ in [illegible] points of a curve $C^?$ of order [illegible] the order of the $n$ is [illegible] by $n(n+1)$ is therefore the rank for the order of the curve $C$, of points in which tangent lines $R^2$ [illegible] [illegible] an $n$ is [illegible] [illegible] the [illegible] of [illegible]. [illegible] [illegible] [illegible] by finding [illegible] [illegible] [illegible] points [illegible] curve $C$ [illegible] a [illegible] plane the points of intersection of $C$ with

a ... plane ... determined by ... and ... a ... plane ... points without degen- ... Eq. ... does ... has ... degenerate. ... plane ... the ... in ... cubic ... rest of ... m(m+1) ... of a pencil that ... in ... fold points of ... cubic ... since ... have ... in ... plane, this ... m(m+1) ... In addition ... in a ... the ... of the ... cubic ... must ... again. ... (c-2) m(m-1) ...

the order of $C_2$ is

$$5m^2(m+1) + (2m-2)m(m+1) = m(m+1)(5m-2).$$

Hence,

Each $K^3$ that has [illegible] with the $m$ in the surface must cut the $m$ in again in a curve $C_2$ of order $m(m+1)(5m-2)$. $C_2$ has $10m$ $m(m+1)$ fixed points at the $10m$ points in which the $K^3$ must cut the $m$-ic.

The cubic of theorem 2) is a special case of this theorem.

Let there exist $F'$, or the $(m+1)$-ic surface cutting out $F'$ in only two ways:

1) At a point of contact of a basic $K^3$ osculating the $m$-ic.

2) At a point of contact of a bitangent basic $K^3$ to the $m$-ic.

In the first case, since by inclusion in the second the

number of points of intersection of the second kind is

$$\tfrac{1}{2}m(m+1)^2(5m-2) - 2m(m+1)(m+2)$$
$$= m(m-1)(m+1)(5m+6).$$

Hence,

where $m = m(m-1)(m+1)(5m+6)$ [illegible] is the degree of a [illegible] is in the case.

The order of $\Sigma$ may be obtained in another way. The locus of such $k^3$'s that meet a given line $l$ is a ruled surface that contains the $k^2$'s [illegible] by $l$. It is [illegible] we have said above that the [illegible] locus was $m(m+1)$ fold [illegible] the locus of $k^3$'s tangent to $\Sigma$ is $\Sigma$ itself. Also two [illegible] lying only in $k^2$'s and in three lines. If $k$ [illegible] the order of the locus of $k^3$'s

basic $K^3$'s,

$$5x - 10m(m+1) \cdot 3x = x \cdot 5m(m+1)$$

Hence

Basic $K^3$'s tangent to $m$ in $x$ form a cone lie on a surface of order $5m(m+1)$.

This surface touches the $m$ in $x$ along $F^7$ and cuts out the curve $C_6$. Hence the order of $C_6$ is

$$5m^2(m+1) - 2m(m+1) = m(m+1)(5m-2).$$

21. If a point $x$ of a variable basic $K^3$ runs along an $m$ in $C_{6m}$ the tangent to this $K^3$ at $x$ generates a ruled surface. The order of this surface is the number of generators cut by an arbitrary line $\pi$. Now the line $\pi$ determines a certain surface which meets the $m$ in $C_{6m}$ in $3m$ points. To the $3m$ and a basic $K^3$ on $m$ there will

three surfaces $\Sigma\Omega\pi$. It follows that $3m$ generators $\gamma_i$ of a ruled surface $\Sigma$ meet $\pi$. Hence

The locus of the tangent $\alpha x$ to the cubic $R^3$ on a point $x$ of an $m$ in $\pi$ is a ruled surface $\Sigma$ of order $3m$.

Since the curve of contact of $\Sigma$ with an $m$ in $\pi$ is of order $m(m+1)$, it follows that

The tangent to $R^3$ through a point $x$ in $\pi$ touches $\Sigma$ at points of contact which lie on a ruled surface of order $3m(m+1)$.

If the $m$ in $\pi$ is a fixed line, then the ruled surface of contact tangents is of order 18. It meets the plane $\pi$ in a curve of order 36. Out of this curve the curve $F'$ of order six must factor twice.

a plane of ordinates $z = f(x, y)$ to be constructed for $M$ is tangent to a tangent $S'$ cone to the quadric and has vertex in the fourth point without loss, centered, on it. It follows that the curve of intersection of the quadric and the circle surface of contact tangent is the curve $F'$ and linearly planar. This is then a finite form in regular points of which there are a curve of twelve lines of cross sections on the quadric in the analytic complex of tangents to curve $K'$.

12. If the $m$ is contact three lines, a, tangent $K''$, three will appear in the curve $F'$. For instance a Middle Surface with nodes at the same point contains the two lines and the $F'$ on the one nodal the curve $F'$ under the two the

two cubics have the 9 in the common, and a residue of order

$$4 \cdot 3 - 10 - 2 = 1.$$

If a residual point of this sort be taken, and that the cubic is the locus of the vertex of a pencil of tangents from the remaining nodes.

If the [illegible] a quadric and the three points, in which it meets the [illegible] be taken for the surface without loss of generality on it, a [illegible] required cubic passes through the $P^7$ and the quadric. It is easy to show that there is actually a cubic $F^7$ in this case. Its locus is

$$\Sigma \alpha_{ij} x_i x_j = 0$$

where $\Sigma \alpha_{ij} = 0$

$F^7$ is cut out by the cubic surface

$$\Sigma \alpha_{12} x_1 x_2 (x_1 + x_2) = 0,$$

which [illegible]

nothing but [illegible] of F being six, we have that there are four basic $R^3$'s on a [illegible] quadric, a theorem of Reye. Given a pencil of basic quadrics

$$\Sigma \alpha_{ij} x_i x_j + \lambda \Sigma \beta_{ij} x_i x_j = 0, \qquad (20)$$

with the conditions

$$\Sigma \alpha_{ij} = \Sigma \beta_{ij} = 0,$$

the basic $R^3$'s on quadrics of this pencil [illegible]

$$\Sigma \alpha_{ij} x_i x_j (x_i + x_j) + \lambda \Sigma \beta_{ij} x_i x_j (x_i + x_j) = 0 \quad (21)$$

Eliminating $\lambda$ from (20) and (21) we have as the locus of basic $R^3$'s on quadrics of the pencil (20) the [illegible]

$$\begin{vmatrix} \Sigma \alpha_{ij} x_i x_j & \Sigma \beta_{ij} x_i x_j \\ \Sigma \alpha_{ij} x_i x_j (x_i + x_j) & \Sigma \beta_{ij} x_i x_j (x_i + x_j) \end{vmatrix} \qquad (22)$$

or

$$\Sigma (\alpha_{12} \beta_{13} - \alpha_{13} \beta_{12}) x_1^2 x_2 x_3 (x_2 + x_3)$$

$$+ \Sigma (\alpha_{12} \beta_{34} - \alpha_{34} \beta_{12}) x_1 x_2 x_3 x_4 (x_1 + x_2 - x_3 - x_4) = 0.$$

this quintic surface must contain the
base curve simply since it does meet the
locus of vertex $K^{-1}$ resulting in (22) only from terms $K^{-1}$, the $K^{(1)}$ on $K^{(2)}$ which
passes through the line segment (22) and
hence the base curve, must be con-
tained in it to order 2 at the [illegible] intersection of order 23 of the two surfaces
(since in $K^{(2)}$ segment meets (22)
without being contained in it, hence
the further intersections of locus $K^{-1}$
with the surface must all be at
base points. The base points are, then,
triple points of (23). A generator of
the generating pencil meets (23) in the
two base $K^{-1}$ on the generator and in
the base $E^4$, the base curve of the pencil.
All of these facts may be known
a priori from the equation of the
surface.

13. The locus of the cubic curves of contact of basic $K^3$'s with quadrics of the general pencil

$$(\alpha x)^2 + \lambda(\beta x)^2 = 0$$

is the quintic surface

$$\begin{vmatrix} (\alpha x)^2 & (\beta x)^2 \\ (\alpha x)(\alpha x^2) & (\beta x)(\beta x^2) \end{vmatrix} = 0 \qquad (13)$$

(13) is on the nine points $(\alpha\beta)$ and contains the $K^3$ defined by the general set of seven through nine of the nine points with the nine tangent planes at the remaining point of the pencil containing the point. It meets a $K^3$ in (21) in 15 points; nine are on the nine points, and the other six points are on the six points of the locus of points of the pencil on the $K^3$ that have one fixed tangent plane. The nine points is evident from the fact that the equation of the pencil and one of

on $K^3$ a binary sextic of the form

$$(\alpha\beta)^6 + \lambda(\beta\delta)^6 = 0,$$

and the discriminant of the binary sextic is of degree 6.

24. The locus of basic $K^3$'s meeting a line in a special pair of $(22)$; in this case the pencil of quadrics on the line would be binomial [illegible]. In this case the equation of the surface is of the form. A pencil of conic quadrics on the line joining the points $a$ and $b$ may be written

$$\sum \alpha_{ij}\, x_i x_j = 0$$

with the conditions,

$$\sum \alpha_{ij} = 0$$

$$\sum \alpha_{ij}\, a_i a_j = 0$$

$$\sum \alpha_{ij}\, b_i b_j = 0$$

$$\sum \alpha_{ij}(a_i b_j + a_j b_i) = 0$$

The two basic $K^3$'s on a quadric of the pencil [illegible]

Eliminating $x_{ij}$ from these six equations we have the Quintic Surface in the form of a six rows determinant:

$$|x_i x_j,\ x_i x_j(x_i + x_j),\ 1,\ a_i a_j,\ b_i b_j,\ a_i b_j + a_j b_i| = 0 \quad (4)$$

25. The argument which we have used to show that the order of the locus of basic $K^3$'s tangent to an $m$-ic surface is $5m(m+1)$ may be used to show that the order of the locus of basic $K^3$'s meeting an $m$-ic curve is $5m$. The same argument shows that the order of a locus of basic $K^3$'s is completely determined by the multiplicity of the given curves in the surface. If the multiplicity of these lines is $m$ the order of the surface is $5m$. If the $m$-ic above is on the base point the number of points upon it

points in which it meets the base plane is reduced by three, the multiplicity of the base lines in the corresponding surface is reduced by three, and the order of the locus of basic $K^3$'s meeting the curve is reduced by 15. The order of the locus of basic $K^3$'s meeting a basic $E^4$ is then

$$5 \cdot 4 - 15 = 5.$$

[illegible] on the $E^4$ meet one of these $K^3$'s in six points and hence [illegible] contain it entirely. An $K^3$ on a quadric meets an $E^4$ on the quadric in six points, the points in which it meets the plane cutting out the $E^4$. Hence the theorem,

The surface (VII) may be considered as the locus of lines $K^1$ meeting the curve $E^4$

a point of [illegible] on the [illegible] curve
[illegible] the [illegible] of the
[illegible].

If we [illegible] four [illegible] fixed
points on the [illegible] of the [illegible]
of rank $k^{(2)}$ [illegible] the [illegible] is
a surface of order $5n - 12p$, having
the line $(n - p)$ [illegible] $(n - p)$ fold line,
etc.

(2) Thus

the locus of [illegible] $k$ [illegible]
on $k^{(2)}$ on four of the [illegible] lines joining
[illegible] a four-nodal cubic surface
with nodes on the four fixed points
and on the [illegible] line joining.

4b. The section of the surface (22) by
a plane $\alpha$ is a curve of interest. We
approach this curve from another point
of view. By theorem f) a basic $E^4$
must any plane $\alpha$ in a set of

four points apolar to $C_a$. Every conic of the pencil $\Phi$ on these four points is apolar to $C_a$ these conics are the sections by a of basic quadrics on the cubic $E^4$. There are two sets of 3 points $i_3^{(3)}$ on each conic of the pencil, since there are two basic $R^3$'s on a basic quadric. The locus of sets $i_3^{(3)}$ on conics of the pencil $\Phi$ is met by one of these conics in the base points of the pencil and in the six points of the two sets $i_3^{(1)}$ — ten points in all. This shows that the locus is a [illegible], $R^{10}$, $\Phi$ is on the ten points of the configuration $B$ in $\pi$. For consider the conic $\Phi$ on 12. This conic being apolar to $C_a$ cuts the polar line of 12 in a pair of points apolar to $C_a$. But 12 taken with any two points of the polar line [illegible] which are apolar to $C_a$ [illegible]

The quintic Q is on the ten points of the configuration $\Phi$ in $S_3$ and so is one of the $\infty^3$ of this configuration on these ten configuration points and on two further points $\phi$ of $C_x$.

It may be pointed out in passing that these two further intersections form a 2-point $i_2^{(3)}$, an determinantal of the pencil $\Phi$.

3 of the 5 points $i_5^{(3)}$ lie apart to $C_x$, on the other three.

The lines joining 3 points $i_5^{(3)}$ in pairs envelope a conic $\Phi$ of a pencil $\Phi$ to touch a curve, $K$, of class three:

For $K$ is the polar reciprocal of the quintic $Q$ in the conic $C_x$.

This gives the further theorem:

Between a line $k$ of $K$ and $X^3$ which lie on quadrics of a pencil

pencil are in a quintic scroll $S_5$.

There are $\infty^1$ of these lines on $F^3$ and on $S_5$, a bicount $\beta$ on $K^3$. The lines constructed in this way are all distinct since there is only one $K^3$ through a given line. Hence they lie in a complex, and the fact that $K$ is the complex curve from the plane is given on the above theorem.

For a line $\alpha$ would, in general, would require the surface $(11)$ to have a double curve, which is not true. Hence,

*The curves $\Phi$ and $\Sigma$ are quintics of genus six.*

If the pencil $\Phi$ is a line in latin with the pencil of lines on its plane, $M$, as $C_2$, the pencil $\Sigma$ becomes a $\Sigma$ latin with a pencil with node at $M$. This gives

Now ... the section by a of the Steiner surface at the nodes of 1, 2, 3, 4, M. Similarly the curve K reduces to the points M and a line meeting it with M for a double tangent.

27. Since K is the proper transform of Q as to Φ, we have,

K touches the fourteen lines of the base points of Φ. It also touches the lines of the configuration B.

Calling the four base points of Φ a, b, c, d, a degenerate conic ab cd must contain two 3-points $v_3^{(1)}$. Each line ab and cd must contain two points of a set, since every line has a unique bisecant basic $R^3$ K

K touches the six lines joining the base points of Φ.

It follows that

Q passes through the ...

tices of the 4-line which is [illegible] reciprocal as to $C_\alpha$ of the base 4 point of $\Phi$.

28. We have mentioned the duals of figures associated with a configuration $\Gamma$. The 3 point $x^{(1)}$ (and) the 3-line on these 3 points are dual figures. These set of three lines are determined by an arbitrary 4-line of $\Pi$ in exactly the same way in which the $\Gamma'$ was determined by an arbitrary point. There are two point determined on any line of the plane the two points of intersection of the director conic $K^2$. Dually, two lines are determined on any point of the plane the two lines to the points of intersection with the conic of the director $K^2$ on that point. There sets of points and lines, reciprocal [illegible]

completely, and in dual ways, by the
configuration K without reference
to the ways in which it can be
[illegible] Thus K may be regarded
[illegible] as the locus
[illegible] and by a net of the quad-
[illegible] projective [illegible] from a
[illegible] between C and K is,
[illegible] completely dual. K may
be regarded as the locus of point-
sets $\iota_3^{(\infty)}$ inscribed in conics of a
[illegible] or as the locus
[illegible] of vertices of line-sets circum-
[illegible] a [illegible] range of conics
[illegible] Similarly K may
be regarded as the locus of line-
sets inscribed in a pencil of conics
apolar to $C_\alpha$ or as the locus of line-
sets circumscribed about a range
of conics apolar to $C_\alpha$. The pencil

and range are polar reciprocals. It is known that for a line in general in a cubic (such a curve and the like) a conic of the pencil determines in this way a conic of the range, and vice versa. Similarly we have the theorem,

If a conic (in line) in apolar to $C_x$ there are four lines such that the six vertices of these 3 lines is apolar (in point) to $C_x$.

From what we have said we have,

The curve Q meets the curve $\overline{K}$ in pairs of points apolar to $C_x$ and hence is

The following theorem is evident.

The Hessian [illegible]

line on [illegible] a pair of points which are the [illegible] points of the points in which the line meets $C_2$ and the points in which the line is [illegible] by [illegible] on $M_1$, $M_2$ or $M_3$. To [illegible] points of the [illegible] $P_1$, $P_2$ and $P_3$ we [illegible] [illegible] [illegible] points.

The [illegible] point of $P$ a 4 point [illegible] $C_2$, determine a second [illegible] figuration $P'$ on $C_2$. We may state the [illegible] we have found above in another way:

A second configuration $P'$ which the same polarity $C_2$ as $P$ determines [illegible] points $R_1$, $R_2$, $R_3$, a point pair $R$ and a line pair $K$; [illegible] $P'$ and $C_2$. $C_2$ determines the point of $P$ configuration, and [illegible] the [illegible] of each configuration [illegible]

6 points adjacent to $C_{\alpha,\beta}$ - sets $i_2^{(3)}$ are determined by the other configuration.

It will be noticed that the two configurations B and B' bear exactly symmetrical relations to the quintics C and K, the quintic Q and C' determined by considering B or B' as fundamental and the lines of the two configurations in the same points, - six-ty points, - and these two quintics must therefore coincide. Similarly B and B' play symmetrical rôles with regard to K.

30. Q, being entirely symmetrical with regard to both configurations, now, a quality will be generated by combining any other conic with 4 points of the configuration B' whose two arithmetic 4-point determinants lie [illegible]

point of the two four-points. The first $E^1$ is on the surface (22). But the second must also lie on (22). For the two $E^1$ determine a general surface which must be in the ruling $Q$. Hence any two cubic $\Phi$ curves of $\Pi$ determine the same quintic curve $Q$. The two quintic surfaces have the curve in common, and hence must contain one fourth $E^1$ coming from each, that is, the $E^1$ must coincide.

We must show that the new $E^1$ must be an $E^1$ on a point on an arbitrary plane $\pi$ between these two. There must be an infinity of such $E^1$ on the surface (22). This may be in another way than this is true. Any two such $E^1$'s lie on one and only one basic quintic. Hence any two basic $E^1$'s on (22) that do not lie on a quintic

of the generating pencil. These $R^3$'s determine a basic quadric which meets (22) in the two $R^3$'s and in a quartic curve basic since (22) has triple points at these points. This quartic is a basic $E^4$. For an $E^n$ on a quadric cuts the generators of both systems $n$ times; an $R^4$ on a quadric cuts the generators of one system three times and of the other once. For first, the $n$-ic quadric on an $R^4$ is the locus of lines $l$ which meet the $R^4$ in $n$ points. Each generator of the quadric above meets (22) in five points. It meets the pair of $R^3$'s in three points, the points in which it meets the $R^4$ being free cutting them out. Hence the residual quartic meets every generator of both systems of the cutting quadric twice and [illegible]

$E^4$ &c.

The surface (22) contains ∞ⁿ pairs of basic $E^4$'s.

Again, any basic quadric on an $E^4$ of (22) cuts out the $E^4$ and a Septic with five actual nodes. A Septic in space with five nodes projects from any one of these nodes into a plane quartic with four nodes, that is, it is projected from any one of its nodes by a pair of quadric cones. The septic must then break up into two $R^3$'s on the five nodes. We see then that any quadric on a basic $E^4$ of (22) cuts out a pair of basic $R^3$'s. Here are $\infty^2$ quadrics on pairs of basic $R^3$'s of (22), and ∞ quadrics on each basic $E^4$. Hence,

The system of basic $E^4$'s

is ∞ ly ∞ infinite.

The following theorems are also obvious from what we have said,

(21) The locus of the $K_1$ on quartics of the pencil on any double $E^4$ of the surface.

Two $E^4$'s of the system on (22) have one and only one quartet in common, and there are two of these $E^4$'s through any point of (22).

On the plane $\pi$,

there are $\alpha'$ configurations $\beta_1$ indistinct — $\Omega$ and $\Omega'$ having the same peculiarity $C_0$.

The locus of the points of these configurations is the curve $K_1$ the polar conjugate of $C_0$ with $C_1$.

Any quartic curve passing through the $\beta_1$ of the first configuration is a locus of these $K_1$'s

$K^3$ meeting the surface in one point not a base point must lie on the surface. The argument used above applies throughout to such a surface and we have,

The surface $(\chi 2)$ is the general ~~point~~ surface of order $\chi 2$ with $\chi 2$ double points.

31. Let us apply the facts of the previous section to the special surface $(24)$, the locus of basic $K^3$'s meeting a line $p$. Basic quadrics meet the surface in $(24)^2$ the fundamental $K^3$ to $p$ in one point only, apart from base points. Basic quadrics on pairs of base $K^3$'s of the surface cut out of the surface a system of moving curves. The line $p$ must be a tangent of the surface on the base points of the $K^3$'s on it and hence there are

the quartic section of (24) by a plane on $\phi$ is identical with a section of a Weddle surface through 8 nodes. The ten [illegible] in the section [illegible] in the quartic, [illegible] quartic [illegible] from the [illegible] to the

The quartic surface of a [illegible] through 8 nodes [illegible] section $\phi$, also having the [illegible]. This [illegible] is defined by the [illegible] of the [illegible] of tangency of tangents from the node

32. It is easy to see that each $K^3$ of the $n(n+1)(n+2)$ points of intersection with an $(n+1)$-ic and an $(n+2)$-ic surface, say $f'$ and $f''$ respectively. The locus of points of osculation of basic $K^3$'s with surfaces of the net

$$f + \lambda\varphi + \mu\psi = 0$$

is, by elimination of $\lambda$ and $\mu$ from this equation and

$$f' + \lambda\varphi' + \mu\psi' = 0$$
$$f'' + \lambda\varphi'' + \mu\psi'' = 0.$$

Its equation is,

$$\begin{vmatrix} f & \varphi & \psi \\ f' & \varphi' & \psi' \\ f'' & \varphi'' & \psi'' \end{vmatrix} = 0, \qquad (25)$$

a surface of order $n + n+1 + n+2$ or $3(n+1)$ times,

[illegible] $K^3$'s osculate surfaces of the net in points of

a surface of order $(2n+1)$.

In particular,

Point of osculation of $2n$ $R^{n-1}$'s with planes on a point lie on a cubic surface.

This surface evidently contains the point singly. This may be seen geometrically or from [illegible]

33. A "higher null system" in space has three characteristics: $\alpha$, the number of null planes on a point; $\beta$, the number of null-points on a plane; $\gamma$, the number of lines each null-point and null-plane are incident with a given line. Basis $R^{n-1}$'s determine such a null system: the null points of a plane are points of osculation of $2n$ $R^{n-1}$'s [illegible]

Steam from [illegible]

The osculating plane at $x_1(t=0)$, is

$$\xi_1 = x_2 x_3 x_4 (x_1 - x_2)(x_1 - x_4)(x_4 - x_1)$$

$$\xi_2 = -x_1 x_3 x_4 (x_1 - x_3)(x_3 - x_4)(x_4 - x_1)$$

$$\xi_3 = x_1 x_2 x_4 (x_1 - x_2)(x_2 - x_4)(x_4 - x_1)$$

$$\xi_4 = -x_1 x_2 x_3 (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$$

The locus of points of osculation of basic $K^3$'s with planes of a sheaf on a point $k$ is

$$k_1 \xi_1 + k_2 \xi_2 + k_3 \xi_3 + k_4 \xi_4 = 0 \qquad (27)$$

where the $\xi$'s have the values given in (26). (27) has triple points at the base points and contains the ten [illegible] lines, like

$$x_1 = 0 \qquad x_2 - x_3 = 0$$

These are the other fifteen lines of intersection of the ten basepoints [illegible] which do

foci of degenerate conics $K'_1$, etc. if it is geometrically evident that they cannot lie on (27).

Since some plane $\pi_1$ is a tangent to an $R^2$ at the osculating plane at the point of tangency, we have,

(27) Contains the asymptotic curve of osculation of tangent from $x$ to the Weddle surface with nodes at $1, 2, 3, 4, 5, K$.

Then

(28) touches the Weddle surface along the locus of osculation.

For it cuts the Weddle surface in a curve of order $4 \cdot 6 = 24$. Out of these curves the ten base lines (factor $\Delta_{10}$) and the ten lines $\pi_{ik}$ must be the tangents to the asymptotic curves

since the Weddle surface is the locus of points of intersection with lines on $\infty$ of the bisecant quadric $K''$, and as lines on $K''$ are of the first order, an osculating plane.

35. In a higher null system with characteristics $\alpha\beta\gamma$, the locus of null-points on planes of a pencil is a curve of order $\beta+\gamma$ meeting the axis of the pencil $\gamma$ times. Since,

Basic $K^3$'s osculate planes of a pencil on a line $\pi$ in points of a curve of order 11 meeting $\pi$ five times.

This 11-ic is the partial intersection of two sextic surfaces (27) determined by ten points of $\pi$ since surfaces must besides on the six lines of intersection of three planes

lies on the cubic surface (16) determined by $\pi$. Any cubic surface which contains a line $\pi$ on $K$ meets (27) in the 11-ic curve cut out by $\pi$ and $K^2$ in the complete system of curves of order six passing twice through the 10 double points.

36. If we require that the points of intersection of the variable quartic $K^2$ with a line $p_{ik}$ be apolar to a ternary cubic form

$$(\xi x)^3 = 0,$$

the line will lie in a complex which is cut out on $\Omega$ by the pencil of planes on the line by $\alpha + \lambda\beta$ and the range of points on the line by $\alpha + \rho\beta$. The points of intersection with $p$ of the variable quartic $K^2$ are given by the quadratic in $\rho$:

$$\Sigma a_{ik} p_{ik}\pi_{il} + 2(a_{ik}\alpha_k + a_{ik}\beta_k) p_{ik}\pi_{ik}\rho + \Sigma b_{ik} p_{ik}\pi_{ik}\rho^2 = 0$$

The points of intersection with $\gamma$ are given by,

$$(\gamma a)^2 + 2(\gamma a)(\gamma b)\mu + (\gamma b)^2\mu^2 = 0.$$

The condition that one of these two quadratics is,

$$(\gamma a)^2 \sum a_1 a_2 p_{12}\pi_{12} - (\gamma a)(\gamma b)\sum(a_1 b_2 + a_2 b_1)p_{12}\pi_{12}$$
$$+ (\gamma a)^2 \sum b_1 b_2 p_{12}\pi_{12} = 0,$$

or, by writing

$$a_i b_j - a_j b_i = p_{ij}$$

$$\sum_{1}^{6} \left[ \gamma_{12} p_{13} p_{23} + \gamma_{14} p_{14} p_{24} - p_{12}(\gamma_{12} p_{12} + \gamma_{13} p_{13} + \gamma_{14} p_{14} + \gamma_{23} p_{32} + \gamma_{24} p_{42}) + \gamma_{34}(p_{13} p_{24} + p_{14} p_{23}) \right] p_{12}\pi_{12} = 0 \quad (28)$$

Hence,

The [illegible] of the second basis $K^2 3$ is in the form of point equations for a given quadric lies in a [illegible] complex.

A special case of this complex is the complex of lines bi-secant to [illegible] and lying

## IV

### Basic Normal Curves in Four Dimensions.

3. The basis in a space of four dimensions is a set of six points. Since the set of six points has $\infty^{24}$ birational quartics on them. We shall call quadric-spreads of the linear 8-fold system on the six points basic quadric spreads. The theorems a), b), c) of part III. may be modified for the four dimensional case as follows:

a') Basic quadric spreads meet any space $S_3$ in quadric surfaces apolar to a definite quadric $Q_{S_3}$ in $S_3$.

c') Basic $R^{(4)}_3$ meet $S_3$ in 4 points $L^{(2)}_4$ apolar to $Q_{S_3}$

Loria, Loc. cit.

Cl Basis $N^{4}$ Conjugate [illegible] Common as points of [illegible].

[illegible] is the [illegible] with regard to which the $(15_6, 20_3)$ configuration, I say, cut out of the complete six point by $\alpha$ is self-polar. As in part III we name the points of the base 1, 2, 3, 4, 5, 6; we have then fifteen base lines 12, etc.; twenty base planes 123, etc. fifteen base spaces 1234, etc. We name the point planes and lines of I' after the line planes and spaces determining them.

31. As before, a [illegible] consequence of theorem 6' —

There is one and only one basis $N^{4}$ which meets a given plane $\pi$ in three lines.

Let $K''$ be the locus of the basis

of $\pi$ as to the quadrics $Q_{\alpha+\lambda\beta}$ on spaces of the pencil $\alpha+\lambda\beta$ containing $\pi$. We shall call it the basic $R^{n-1}$ transcendent to $\pi$.

39. We shall make some use of degenerate basic $R^n$'s, and it will be well to point out how they may degenerate. Any conic $12$ taken with an $R^3$ on $3,4,5,6$, and $\binom{12}{3456}$, the point where $12$ meets $3456$, is a degenerate basic $R^4$. Thus we have pencils of conics that may be regarded as parts of degenerate basic $R^4$'s. The two planes $123$ and $456$ have a point $\binom{123}{456}$ in common. Any conic on $1,2,3,\binom{123}{456}$, taken with a conic on $4,5,6,\binom{123}{456}$, is a degenerate basic $R^4$.

40. We shall say that an $R^3$ is orthic to a quadric $Q$ when every quadric on the $R^3$ is apolar

to Q, or which amounts to the same thing, when there is one 4 point, and hence $\infty^1$, inscribed in the $R^3$ and apolar to the quadric.[1] Since an $R^3$ apolar to Q, any 4 of these 4 points may be connected by, choosing any point a of the $R^3$ and taking with a the three points of intersection with the $R^3$ of the polar planes of a as to Q. So is there conditions on an $R^3$ to be apolar to a quadric, since it is three conditions on the 3-point in the polar plane of a to be [illegible] of the [illegible]

40. Project the system of basic [illegible] from a [illegible] point [illegible] to the [illegible] a system of [illegible] [illegible] [illegible]

[1] Segre, Etoom dei Lincei, [illegible] II, p. 226.

simple infinity of cones $K^4$'s, each through two, points of the cone. For a basis $K^4$ cannot have a cone of these cones in a point of it without having entirely on it, since projection from 1, 2, 3, 4 [illegible] gives an $K^3$ with, as an intersection with a quadric cone. It follows,

There are $\infty^2$ such cubic cones projecting basis $K^4$'s from a point 6. These cones must be in a basis system of $K^3$'s on 16, 26, 36, 46, 56.

11. If on a quadric on five points is polar to a quadric Q, we may call the set of five points an orthic five points on Q. To define characteristic of such a set of five points is that the polar line on Q of the line joining any

two of the [illegible] the plane passing
the other three, from the point
[illegible] and the
conic passing through [illegible] is 34,
which is on 36 46 56, or 3456. 16, 26,
36 are on one line, the double [illegible]
point on $C_4$ and each quadric
on them is cut by $C_4$ in $B_{ik}$. The points
that carry $A^{(1)}$ on them is a conic
$B\Omega_4$. This [illegible] from the above
theorem.

4 points $C_4^{(1)}$ on a conic 4 points
in a conic in $A^{(1)}$ of the Bonic system
on 16, 26, 36, 46 &c, and after $C_4$
$\Omega_5$.

Any $C_4^{(4)}$ is a unique [illegible]
[illegible] was one point [illegible]
by drawing the [illegible] 56,
56 on [illegible] and getting all those on
[illegible] with the [illegible]

of the [illegible] forms [illegible]

there

Since $R^3$'s contain 3 in points of a cubic curve $F$, the curves of pencil of $R^3$'s on 16, 26, 36, 46, 56 with $Q_{ik}$

Since $R^{3}$'s hyp'd-osculate at these 4 points (counted) in points of intersection of these basic $R^3$ with $Q_{ik}$.

We shall call this set of twenty-four points the points $H$.

$R^{3}$'s on the six cubic curves of $T$ define the same curve $F$ and the same twenty-four points of osculation.

For there are six other fivepoints in $T$, which we may call the set 1, 2, 3, 4, 5, 6. Into $\binom{6}{4}$ = 15 quadrics we eliminate by $R^3$ of any one of the six points.

Let us [illegible]

of the [illegible] of reference the
five points 1, 2, 3, 4, 5 with 6 as (1, 1, 1, 1),
[illegible] from [illegible] 1.

Basic $R^4$'s [illegible] is [illegible]
in points of its $m(m+1)(m+2)$ way
[illegible] and a spread of order
$m+1$.

[illegible] osculate a $m$-
spread in points of a [illegible] of
order $(m+1)(m+2)$, the [illegible] of
which [illegible] $(m+1)$-ic
and $(m+2)$-ic spread.

Basic $R^4$'s hyperosculate a
$m$-ic spread in its $m(m+1)(m+2)(m+3)$
points of intersection with a $(m+1)$-ic
an $(m+2)$-ic and an $(m+3)$-ic spread.

In particular,

Basic $R^4$'s [illegible] a [illegible]
a in points of the quartic [illegible] of
order 6.

$$(\alpha x^2) = 0, \qquad (1)$$

(1) being the equation of the polar conic of the point $\alpha$ with respect to the conic $C_{xx}$.

Basic K's osculate $\alpha$ in the points of intersection of $C_{xx}$ and

$$(\alpha x^3) = 0$$

$\alpha$ is a point $F'$ of order $n$ in the complete intersection of a plane curve of order $n$ and a curve of order $n+1$.

Hence $F'$ is represented on the curve by $n$ points of the group of $F'$.

$$(\alpha x^4) = 0$$

[illegible] the conic passing through the [illegible] of the quadrangle $\alpha$ [illegible] from $\alpha$.

The conic $1'$ is therefore one of your new forms for the invariant of the hyperbolic [illegible] of the type of the Poncelet, Hesp. Graffian Hessian, etc. p. 216

43. Several cases in a manner
those we have [illegible] since the
of [illegible] to us.

A. [illegible] for the coord.
of the [illegible] point of inter-
section with a of [illegible]
satisfying a.

B. [illegible] for the locus
of the points in which any
[illegible] tangent to a
[illegible] will of course lie on
$Q_a$.

C. [illegible] for the locus
of the two [illegible] of inter-
section with a of [illegible] lines
equal to a. C will be a surface;
A and B will be twisted curves
in it.

44. [illegible] of A is usually
determined by finding its intersec-

[illegible] a plane [illegible]
section Γ. A basic $R^4$ cannot meet
a base space, and hence cannot
meet a plane of $\Gamma'$ [illegible]
[illegible]. The $R^4$'s [illegible]
cones cannot be made to osculate
x. Consider the $R^4$'s made up of
the lines $12$ and $R^3$'s on
$3$ [illegible] $(12)$. [illegible]
[illegible] $R^4$'s [illegible]
$\Gamma'$, the point $12$ counts as a
quintuple point of $\alpha$. Six points $12$
[illegible] $\Gamma'$. We

The curve A is of order 36.

A is of genus four, since
it is in one-one correspondence
with $\Gamma'$. [illegible]
of points [illegible]
[illegible]

fewer points H.

45 The order of B may be determined in a similar way. Thus, there were $N_1$ such hyperoctahedral faces of $\Gamma$ passing through $P_1$ and $P_2$ can be made to touch a line $a_{12}$ joining a pair of $\Gamma$ vertices of such faces. Each of the opposite cells of $\Gamma$ at $P_1$ and $P_2$. Each pair of such cells cut in two $\Gamma^{(3)}$ cells passing through $a_{12}$. Thus there are two vertices of $\Gamma'$ there being four lines on each pair of $\Gamma^{(3)}$ cells.

The order is of order 16

We now show that B actually breaks down into two octahedra. Considered in $S_{21}$, B is the locus of points of intersection with lines of $R_{21}$ of [illegible]

the locus of points of intersection with lines of a system of three-space $\Lambda^1$'s — the base points, 1, 2, 3, 4, 5, being not specially related to the generation of which the system of lines — is an octavic on the regulus. For consider which this curve can meet the plane 123. 95 meets the system in two points; 5 cones of them correspond to a point in 123. Since it is touched by the common two-fold points of the cones on each of 12, 23, 31. Hence the curve meets 123 in eight points, and is an octavic. It meets a tangent plane to the quadric in eight points; two on one generator and six on the other. It projects from the points of the quadric into a plane octavic curve with one six-fold point and two

double point. Hence it has sixteen apparent nodes, and its genus is five.

B, then, is two octavics; the [illegible] corresponding to the two sets of [illegible] on $Q_\alpha$. Consider the two octavics projected from a point of $Q_\alpha$. The 32 apparent double points of the two curves together with their 64 intersections real or apparent make up the necessary number of double points. There are 2·6·2 apparent intersections along the two [illegible] through the point of $Q_\alpha$. [illegible] intersections are real; [illegible] line of $\Gamma'$.

B [illegible] $(\alpha x^1)$ [illegible] the curve $F^{(\alpha)}$ [illegible] only in the 24 points $\Phi$. Hence a [illegible] $(\alpha x^1)$ and therefore $\Gamma'$ [illegible] these points.

One of the [illegible] pairing up [illegible] with [illegible] the point of contact of [illegible] by [illegible] Suppose [illegible]

$$x_i \quad , \quad x_i t$$

[illegible] at $t_1$, [illegible] $t_2$. Then [illegible] the following [illegible]:

$$\begin{aligned} & \frac{x_i}{1 - k_i t_1} - \\ & \frac{x_i}{1 - k_i t_2} - \end{aligned} \qquad (1)$$

$$\begin{aligned} & \frac{x_i}{(1 - k_i t_1)} = 0 \\ & \frac{x_i}{(1 - k_i t_2)} \end{aligned}$$

The coördinates of any point on the line, passing through the two points of contact, may be written

$$y_i \quad \frac{x_i}{1 - k_i t_1} + \lambda \frac{x_i}{1 - k_i t_2}$$

for

$$(xy^2) = \sum \kappa_i \left( \frac{\lambda_1}{1-\kappa_i t_1} + \frac{\lambda_2}{1-\kappa_i t_2} \right)^2$$

$$= \sum \frac{\kappa_i^2}{(1-\kappa_i t_1)^2} + \lambda \sum_{t_1 t_2} \left( \frac{\kappa_i}{1-\kappa_i t_1} \cdot \frac{\kappa_i}{1-\kappa_i t_2} \right) + \lambda^2 \sum \frac{\kappa_i^2}{(1-\kappa_i t_2)^2}$$

This vanishes for all values of $\lambda$ on account of the condition (2). Since

the lines joining corresponding points of the curves lie entirely in the quadric $R_2$.

which is, we see from above, [illegible] from $y$ on the curve $C$ [illegible]

Corresponding points of $B$ coincide at points where $C$ and $C'$ hyperosculate. Since $B$ touches $F'$ at these points we have,

the points $H$ are points of $F'$ such that the tangents to $F'$ at these points are generators of $C_{2n}$, that is, the [illegible] points of [illegible]

may F' cuts it.

46. We find the order of the surface $C^{14}$ in a similar way, namely, by finding the order of the curve in which it meets a plane of $T^3$, say 1234. The $K^{15}$'s on five points in the space 1,2,3,4 touch the plane 1234 along a conic and cut again along a quintic which is on $T^3$ [illegible] on a conic on 1,2,3,$\binom{123}{456}$, and so on with either of the two conics on 4,5,6,$\binom{123}{456}$, satisfying our conditions. Hence,

The surface $C^{14}$ is of order 14.

It [illegible] the curves B.

F' and cuts out the curves B.

47. The 14-ic $C^{14}$ is analogous to the Steiner [illegible]

case of theorem 4, page III. We may find its equation referred to an orthic base of $T^7$ by a method similar to that used in finding the equation of the conic. Take as base 16, 26, 36, 46, 56 as base. Take as reference points 16, 26, 36, 46, with 56 as (1,1,1). A basic quadric may be written

$$\sum a_{ij} x_i x_j = 0 \qquad (3)$$

with the condition

$$\sum a_{ij} = 0 \qquad (4)$$

The quadric $Q_\lambda$ passing through the points is of the form

$$\sum a_i \xi_i^2 + 2\lambda \sum \xi_i \xi_j = 0, \qquad (5)$$

the $\xi$'s being, as before, $x_i + x_j$, (5) being merely (4) multiplied by $\lambda$. (5), in point is

$$Q \equiv \sum \left[ 2\lambda^3 - \lambda^2(a_2+a_3+a_4) + a_2 a_3 a_4 \right] x_1^2 - 2 \sum \lambda(\lambda - a_3)(\lambda - a_4)\, x_1 x_2 = 0 \qquad (6)$$

If we find the condition that

the polar plane of a point $\xi$ w.r.t. $Q$ touches the curve $K^2$ in a conic which evidently has the equation of $K^2$ when $x_i$ is replaced by

$$\frac{x_i}{1 - x_i t}$$

This meets the polar plane of $\xi$ w.r.t. $Q$ at the points given by the roots of the cubic in $t$

$$\Sigma \frac{x_i}{1 - x_i t} \frac{\partial Q}{\partial x_i} = 0$$

or

$$x_1 x_2 x_3 x_4 \Sigma \frac{\partial Q}{\partial x_1} t^3 - \Sigma x_1 x_2 x_3 \left( \frac{\partial Q}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial Q}{\partial x_3} \right) t^2$$
$$+ \Sigma x_1 x_2 \left( \frac{\partial Q}{\partial x_1} + \frac{\partial Q}{\partial x_2} \right) t - \xi Q = 0 \qquad (7)$$

Now

$$\Sigma x_1 \frac{\partial Q}{\partial x_1} = 2Q$$

(7) may be written

$$x_1 x_2 x_3 x_4 \, t^3 - f_4 t^2 + f_3 t - 2Q = 0, \qquad (8)$$

where $f_3$ is the polar plane of $\xi$ w.r.t. $Q$, and $f_4$ and $f_3$ each vanish a quadratic condition respectively.

(8)

base is $R^1$ of the other three points of the set $(4)$ determined by $x$. The parameter of $x$ is 0. Hence we have,

Basis $R^{11}$ [illegible] $Q = 0$.

Basis $R^{12}$ [illegible] $f_2 = 0$.

Basis $R^{13}$ [illegible]

$f_4 - f_3 = Q = 0$.

[illegible] $f_3$ [illegible] $Q$ [illegible] surface on the curve $F^7$, and $f_4$ a quartic surface cutting out of $F^7$ the [illegible] $H$.

$C$ is the discriminant of (9):

$$C \equiv 108 x_1^2 x_2^2 x_3^2 x_4^2 \Delta^2 Q^2 + 4 x_1 x_2 x_3 x_4 \Delta f_3^2 \quad (9)$$

$$- 36 x_1 x_2 x_3 x_4 \Delta Q f_3 f_4 + f_4^2 Q - f^2 f_4^2 = 0$$

$C$ is of the form

$$\Phi Q + \Psi f_3^2 = 0,$$

and hence touches $Q$ along its intersection with $f_3$. The [illegible] intersection with $Q$ is the curve $\Gamma_{12}$. $C$ is therefore cut out of $Q$ by

$$\Psi \equiv 4x_1x_2x_3x_4\{f_3 - f_4^2 = 0$$

This form shows that B touches T at the point $H_4$ [illegible] all along the curve cut out by $f_4$. The form of $\Psi$ shows also that B has nodes at its intersections with the planes of T.

The curve A must lie on C. We can find from (8) a surface that will cut it out. If x is a point of A (8) must have three equal roots, and its Hessian must vanish identically. This Hessian is

$$q_8 t^2 + q_7 t + q_6 = 0,$$

$$q_8 \equiv 3x_1x_2x_3x_4\{f_3 - f_4^2$$

$$q_7 \equiv 18x_1x_2x_3x_4\{Q - f_3f_4$$

$$q_6 \equiv 6f_4Q - f_3^2.$$

$f_6$ is [illegible]

which are a self-apolar quartic on the basic $R^4$ (or basic $R^3$) determining them. The $f_3$ of this quartic is

$$9 f_3 f_4 Q - 54 x_1 x_2 x_3 x_4 \Delta Q^2 - f_3^3 = 0$$

This is the locus of $C_4^{(4)}$'s that are harmonic [illegible] determining them. It osculates $Q$ along the [illegible] The above equation has the form

$$f_3 g_6 - 3 Q q_7 = 0.$$

[illegible] contains the [illegible] from $q_6$ and $q_7$.

$A$ is the curve common to $q_6$ $q_7$ and $q_8$. $q_6$ and $q_7$ meet in $A$ and the curve $F$. $q_6$ has nodes at the points of the [illegible] planes $x_1, x_2, x_3, x_4, x_5, \Delta$, and $q_7$ has triple points at these points. $A$ is [illegible] see page [illegible]

ble points on the point of $\Gamma'$. $y_2$ and $y_4$ must in $A$ and the tangent common to $f_2$ and $f_4$. $C$ is therefore, by section 2, the cone

$$y_4 y_6 - y_5^2 = 0.$$

$y_6$ then, contains $\Gamma'$ along the curve $\Gamma'$ and $A$. Further, the form of $y_6$ shows that it contains $\Phi$ and hence $C$, along the curve $\Gamma'$.

$C$ may also be put into the form

$$x_1 x_2 x_3 x_4 \Phi + \Psi f_4^2 = 0$$

this shows that $f_4$ meets $C$ at its point of intersection with $x_1 x_2 x_3 x_4$. These six planes form an orthic 6-plane of $\Gamma'$. It may, or may not, happen that $f_4$ has nodes at the ten points of the 6-plane. When the points $H$ are on a [illegible] nodal quartic surface, $\Gamma'$ is

ing the ten lines of any quadric
5 planes of $\Gamma^7$ and each meets in
the ten points of that 5 plane. Thus
on this quadric surface.

The surface $\Phi$ may exist,
that is to say on the ten points
on each of these 5 planes. Hence,

The curve $\Gamma^7$ is on a quadric
surface which has ten points of
each of these 5 planes of $\Gamma^7$

48. We see from (1) that a quadric
$Q_{\alpha+\lambda\beta}$ in a pencil of the pencil

$$(\alpha x)^2 + \lambda(\beta x)^2 = 0$$

cuts out on $\Gamma^7$ by the quadric forms

$$(\alpha x^3) + \lambda(\beta x^3) = 0$$

Eliminating $\lambda$ between these two
equations we have as the equation
of the curve $C_{\alpha+\lambda\beta}$,

$$\begin{vmatrix} (\alpha x^4), & (\beta x^4) \\ (\alpha x), & (\beta x) \end{vmatrix} \equiv \sum \pi_{ij} x_i x_j (x_i - x_j)^2 = 0, \quad (i,j=1,2,3,4)$$

radius $\pi_{ij}$ $(x_i y_i)(x_j y_j)$ are the double
locus of the planes $\pi$ passing to the
system of the points. It is easy
to show that the curve $k^n$ on $x$
passes through $x$ with the tangent

$$p_{ij}^{n-2} x_i^{n-2} x_j^{n-2} (x_i x_j).$$

Jacobian curve.

The locus of points of tangency
with curves $k^n$ of tangent lines must
lie on a plane $\pi$ in the $(n-1)$-space
$(x)$.

In what follows we shall denote
this ground by $|\pi_x|$. We call the plane
$\pi$ the $y^{th}$.

Consider two systems of curves $\varphi^{n-1}$
and $\varphi'^{n-1}$ with $x$ the planes $\pi$ and
$\pi'$ [illegible] $p$ [illegible]
[illegible] intersect on the curve $\Gamma_p$ on
the plane $\pi$ [illegible] $\pi$ meets
$\pi'$ [illegible]

the locus of points of tangency with curves $K^{4'}$ of tangent lines that meet a given line $p$ is a septimic [illegible]

[illegible] is met by a plane $\pi$ on $p$ in $p$ and in six other points on $Q_\alpha$. This sextic curve is easily defined by the basic $K^{3}$'s in $\alpha$. It is the intersection with $Q_\alpha$ of the cubic surface (16), Part III, determined by the line $p$. It follows that the sextic curve meets $p$ twice.

Consider now a line $p$ [illegible] a plane $\pi$ [illegible] a point $a$ [illegible] $[p]$ on $\pi$, [illegible] of $\Delta^{1}$ in [illegible] by the curve [illegible]

on the space $\pi$ [illegible] the residual 15-ic Curve is the locus of points of contact of tangents of the $K^4$ with tangents passing through a fixed point.

The locus of points of contact with curves $K^4$ of tangents passing through a point on the 15 is a [illegible]

[illegible] contains $\pi$ [illegible] points, and [illegible] contains [illegible] each [illegible] each [illegible] curves.

41. There are $\infty^1$ lines tangent to a curve $K^4$ and $\infty^3$ basis $K^4$'s; $\infty^4$ lines in all. Since these lines [illegible] a [illegible] the order of $\Delta^{(4)}$ is 14. that the order of this conjugate is 14, since the conjugate curve cuts on any point a is the curve passing [illegible] from a. [illegible]

are tangents to twice $K^3$'s on the [illegible] of the quadric $Q_x$. Congruences lines meeting a line $p$ are tangents to twice $K^3$'s at points of a sextic curve on $Q_x$. By a theorem of part III these lines lie on a ruled surface of order $6 \cdot 3 = 18$. Now the line of an $(m, n)$ congruence that meets a given line $p$ lie on a ruled surface of order $m + n$. The class of any congruence is therefore four. For,

There are four twice $K^4$'s that touch a given plane $\pi$.

Lines tangent to twice $K^4$'s are in a $(14, 4)$ congruence.

[illegible] meet a line in twelve points. It is known that the lines of an $(m, n, r)$ congruence

that would be true for the general surface of order $m+n$, with $f$ as an $m$-fold line, and that this surface has an additional double curve of order $\frac{n(n-1)}{2} + n$. Call the $(n-1)$-ic surface, or $n-1$ times, and determined by the [illegible] $f$, by a plane $\pi$ [illegible] this plane on the curve or $\pi$ which is in one-one correspondence with the system of contact of conic $K^4$ with the line of our surface, and hence is of genus four. This $(n-1)$-ic has a 14-fold point at $O$ besides

$$\frac{11 \cdot 12}{1 \cdot 2} - \frac{14 \cdot 13}{1 \cdot 2} - 4 = 41$$

nodes. These are the points in which $\pi$ meets the additional double curve of the surface of order $6+n$. Now as

The rank of the congruence of lines tangent to basic $R^4$'s and meeting on any space $\alpha$ is 35.

From what we have said we have the following theorems,

The locus of tangents at points of a line $p$ to basic $R^4$'s meeting $p$ is a cubic 2-way having $p$ for a directrix.

For the line $p$ meets a null-space $[\pi]$ in three points, and the tangents to basic $R^4$'s at these points determine the points in which $\pi$ meets our 2-way.

Also,

the locus of tangents at points of a plane $\pi$ to basic $R^4$'s meeting $\pi$ is a ruled space of order 7.

For $\pi$ meets a section $\sigma$ surface $[p]$ in seven points, and the tangents

to basic $R^4$'s at these points determine the points in which $p$ must cut spread.

50. The points of contact of basic $R^4$'s with spreads of the pencil of surfaces

$$f + \lambda\varphi = 0$$

are cut out by the $(4m+1)$ ic spreads

$$f' + \lambda\varphi' = 0.$$

Eliminating $\lambda$ we have the $(2m+1)$ ic spread

$$\begin{vmatrix} f, & \varphi \\ f', & \varphi' \end{vmatrix} = 0$$

Hence,

Points of contact of basic $R^4$'s with spreads on a pencil of surfaces are on a spread of order $2m+1$.

Similarly,

Points of osculation of basic $R^3$'s with spreads

we have,

$$f + \lambda_1 \varphi + \lambda_2 \psi = 0$$

and so a system of curves $\varphi, \psi$, of the form

$$\begin{vmatrix} f & \varphi & \psi \\ f' & \varphi' & \psi' \\ f'' & \varphi'' & \psi'' \end{vmatrix} \qquad (11)$$

Let us apply this theorem to a particular case. If $f$ is a basic quadric, $\varphi$ and $\psi$ the curves of osculation of basic $K^4$'s with $f$, $\Phi$ say, is a curve of order 24. But a basic $K^4$ having already six points in common with $f$ cannot osculate without lying entirely on $f$. $\Phi$ must therefore be a basic $K^4$. We have then

There exists a basic $K^4$ on a basic quadric $f$ osculated

If $f$ be osculated, $\varphi$ is

of constants, to find a finite number of cubic $K^4$'s on a basic quadric spread: for it is nine conditions on an $K^4$ to lie on a quadric spread and the spread therefore contains $\infty^{11}$ $\alpha''$ $K^4$'s. Three homogeneous remaining degrees of freedom are taken up by requiring the $K^4$ to be on six specified points, since it is two conditions on a curve in a three dimensional space (linear or not) to be on a specified skew pair of $\alpha$'s.

If $f, f_1, \ldots f^\rho$ are $\rho+1$ quadric spreads (11) in the form $\sum_0^\rho a_i f^i$ is $\delta^\rho$. Given any spread of the set

$$\int \varphi \, d\varphi = \psi \quad \ldots$$

(11) is in the form of a sum $f$.

Let us consider spreads

$$f + \lambda\varphi + \mu\psi = 0$$

The curves of the pencil cut the plane $\pi$ in the same points. The spreads of the net cut out on $\pi$ the bi-canonical curves $K'$ of $\pi$. Since this $K'$ meets any of these spreads in $6+3=9$ points. Now a quadric spread in $S_4$ containing a plane must be a cone and contain two systems of planes on the vertex, and $K'$ cuts it in two plane sections of one system (these lines meet the other system once). If the $K'$ on the spread $(11)$, then meets $\pi$ in $9$ points, a line $l$ of $\pi$ meeting $\pi$ once on the $l$ in $3$ on some generatrix of the cone, meeting that generatrix (and each other generatrix) in one other point. Consequently the $K'$ on the spread

of it. Hence,

The locus of basic $K^4$'s meeting a plane $\pi$ in a system of order 9 has the transcendent basic $K^4$ in $\pi$ for a triple curve. Its equation is of the form (11).

It follows that,

There are $\infty^1$ basic $K^4$'s meeting a plane and a line.

Basic $K^4$'s meeting a line lie on a 2-way of order 9.

51. Hence being on $\infty^1$ basic $K^4$'s, there are in general 11 points common to 9, satisfying 11 characteristic conditions. If we can, by means of the knowledge we have of the systems $A$ and $P$ and the equations (2), (3), (4), with the three systems just enumerated, find the number of basic $K^4$'s satisfying 9 conditions of these forms, we shall

We introduce the following symbolical notation. Denote by

$P_6$ the 15 fold condition that a curve $K^4$ be basic,

$P$ the three fold condition that a curve $K^4$ be on a assumed point,

$V$ the two fold condition that a curve $K^4$ meet a given line,

$\mu$ the one fold condition that a curve $K^4$ meet a given plane,

$\nu$ the three fold condition that a curve $K^4$ touch a given plane,

$\alpha_2$ the one fold condition that a curve $K^4$ touch a given space,

$\alpha_1$ the two fold condition that a curve $K^4$ osculate a given space,

$\alpha_4$ the three fold condition

that a $K^4$ lies entirely in a given space.

For a $K^4$ subjected to a $\lambda$ fold condition, a finite number for loci $a(4-\lambda)$ will be found which we shall denote by $[T]$, $[T]$ being the symbol of the system as explained in part III, the symbols $|T|$ and $|T'|$ being used as in part II, to denote $K^3$'s and $K^2$'s in space of order $d$. Certain planes (lines, or curves, etc.) will be found to join the curves of the system $[T]$ in a number of ways of which the $\alpha$ ways apart from base lines, curves or surfaces, and base planes, it is important to know the multiplicity of the base lines, curves and planes in these symbols. The multiplicity of the base points

incidents from the proof that $C_{mn}$ $K^{n}$'s $C_{mn}$ [illegible] except at base points with a typical exception in it. The following [illegible] give the multiplicities of base [illegible] planes and conics in a number of these systems. Accented letters are meant to indicate that the [illegible] repeats upon [illegible] a different element; [illegible] the same element is [illegible]

| System | Dimension of System | Mult. of Curve | Mult. in Plane | Base Conics | $\Sigma \mu^2$ |
|---|---|---|---|---|---|
| $[\nu]$ | 2 | 1 | 0 | 0 | 1 |
| $[\mu]$ | 3 | 3 | 1 | | 9 |
| $[\alpha_2]$ | 3 | 6 | 2 | | 18 |
| $[\alpha'_2]$ | 2 | 6 | 0 | 3 | 54 |
| $[\alpha^2]$ | 2 | 0 | 0 | 2 double conics on each pt | 84 |
| $[\mu^2]$ | 2 | 1 | 0 | 1 conic on each pt. | 11 |
| $[\mu_1']$ | 2 | 3 | 0 | 2 conics on each pt | 61 |

| Species | Dimension | Mult. of 15 & terms | Mult. of 20 & terms | Base Curves | $\nu_1, \nu_2$ |
|---|---|---|---|---|---|
| $[\nu_1 \nu_1']$ | 1 | 10 | 0 | one double curve and 2 simple curves on a pp | 12 |
| $[\nu_2 \nu_2']$ | 2 | 20 | 0 | 4 double curves on each pl. | 244 |

We now verify the facts contained in this table.

We have shown the order and dimension of $[V_\mu]$ to be $\frac{1}{2}\mu(\mu+1)$ and $2^\mu$ respectively. If we take the base lines $1, 2, 3, 4, \binom{56}{1234}$ and the point where the lines 1234, taken with 56 lines on the $\nu$ meet.

We have seen also that $[\mu\nu]$ is a $3^{\nu}$ way of order $\frac{1}{2}\mu(\mu+1)$ whose general curve passes $\nu(\nu+1)$ times through a point, and the curves of the general $\nu$ have $\frac{1}{2}\nu(\nu+1)$ on this point, taken with any one of the $\frac{1}{2}\mu(\mu+1)$ on $456$, a series of $4^{\nu}$ according to

locus of conics covers a plane singly (there is a unique conic on every point of the plane) but nevertheless contains the cross planes singly. $(u)$ and $(v)$ must meet in 9 basic $K^{4}$'s, the 4 $K^{4}$'s meeting $u$ and $v$. They must lie in a curve of order 81. This leaves a curve of order $81 - 36 = 45$ to be accounted for by the fifteen base lines. Hence $(u)$ must contain the base lines triply, since $(v)$ contains them simply.

$(u)$ is a curve of order 15, since $[v]$ meets $Q_{2}$ in 15 points and the basic $K^{4}$'s on these points determine 15 points in which it meets $(u)$. Two cones of the pencil contain $u$ and hence $v$, besides the pencil of $xy$ cones

lowing in this ground $[\alpha_2]$ and $[\nu]$ must in 18 base $\mathfrak{C}^{43}$. They must in a curve of order 162, on which the base lines form a curve of order $162 - 72 = 90$. Hence $[\alpha_2]$ contains the 9 base lines simply. It follows,

The point of the configuration $T'$ in $\alpha$ are multiple points of the $\alpha$-surface $\mathfrak{C}^{43}$.

$[\alpha_3]$ is a curve of order 34, since $[\nu]$ must the quartic $\Gamma'$ in $\alpha$ in 34 points. Via $\Gamma^{33}$ of the base $\alpha$-points in 1234 counted 4. Hence the base lines are six-fold.

$[\alpha_1^2]$ must $\alpha$ in 18 fold base line $\alpha\Omega$. $\alpha_1 \alpha_2$ is therefore 52. Two curves of each of the five cubic $123$ and $4\,5\,6$ consist $\alpha$ and three curves consist double in $[\alpha_1^2]$.

$[\alpha^2]$ [illegible]

singly, since there is a unique $K^3$ on $1,2,3,4, \binom{56}{1234}$ bisecant to the line in which $\pi$ meets $1234$. $\Delta_7$ contains on each plane thru [illegible] the point in which it meets the plane. If $\Delta_7$ [illegible] two lines, by combination with $(\pi)$ $\Delta_7 \cdot \Delta_7' = 4 \cdot 4 = 16$, whereas $= 17$. $[\pi\pi']$ contains the two lines in the plane $\pi\pi'$. There are four quartic $K^3$'s in $1234$ meeting $\pi$ and $\pi'$. $\Delta_7$ contains the two lines meeting the $K^3$ in which $\pi$ meets $\pi'$ [illegible] a line in the plane. If $\Delta_7$ [illegible] by combination with $(\pi)$ $\Delta_7 \cdot \Delta_7' = 4 \cdot 4 = 16$ whereas $= 15$. $[\pi,\pi]$ contains the two lines 10 times, since there are ten $K^3$'s in $1234$ which meet $\pi$ [illegible]

$\alpha$. Two conics on 123 and two on 456 touch $\alpha$. Lines of these tangent conics on 123 taken with the 456-conic on the point in which $\mu$ and $\nu$ is a base $B^{\alpha}$ satisfying the conditions. The latter cannot lie in $\alpha$, else $\mu$ and $\nu$ would be $\alpha$. Combination with $[\mu\nu]$ gives the $\alpha$.

$[\alpha, \alpha']$ contain the base curve $D$ of $\Delta$, since through $D$ are $\infty^3$ $\alpha$ and $\alpha'$ planes. Each $\alpha$ and each $\alpha'$, hence there are four double conics on every plane. Combination with $[\mu\nu]$ gives the $\Delta$.

The following table gives the number of [illegible]

Steiner [illegible]

various 3-fold combinations. Many of these may be obtained by the combination of the different forms given in the foregoing Table, together with the application of the theorem of Sturm. Others are added for the sake of completeness.

| | | |
|---|---|---|
| $P_6 P$ 1 | $P_6 \alpha$ 11 | $P_6 \alpha_2^2 \mu = 32$ |
| $P_6 \mu^3$ 1 | $P_6 \alpha_2 \mu$ 34 | $P_6 \alpha_2^2 \alpha_2' = 64$ |
| $P_6 \rho$ 4 | $P_6 \alpha \alpha'$ 101 | $P_6 \mu \mu' \mu''$ 61 |
| $P_6 \alpha_4$ 4 | $P_6 \mu^2 \mu' = 17$ | $P_6 \alpha_2 \mu \mu' = 122$ |
| $P_6 \alpha \nu$ 9 | $P_6 \mu^2 \alpha_2 = 34$ | $P_6 \alpha_2 \alpha_2' \mu = 244$ |
| | | $P_6 \alpha_2 \alpha_2' \alpha_2'' = 488$ |

The first three rows of these results could be obtained in two ways previously, the last row, by the application of the preceding table.

It may add to the value of the following enumeration of the

terms of part II.

Basic $R^4$'s containing a 2-way of order $m$ lie on a 3-way of order $4m$.

Basic $R^4$'s meeting a curve of order $m$ lie on a 2-way of order $4m$.

Modifications of these theorems will be necessary if the $R^4$'s considered lie on a 2-way.

52. Consider a pencil of spaces

$$(\alpha x) + \lambda(\beta x) = 0,$$

having a plane $\Delta$ as axis. The general space meeting both the quadrics $\Omega_1$ and $\Omega_2$ in the pencil $\alpha + \lambda\beta = 0$

$$(\alpha x) + \lambda(\beta x) = 0.$$

It follows that the pencil $\Omega_1 + \lambda\Omega_2 = 0$ cuts on $\Delta$ a pencil of curves. Now, the base points of a pencil

with S of its tricoaxal $\alpha$ is in $K^4$ are apolar to all quadrics $Q_{\alpha+\lambda\beta}$ by (theorem c'). They must, then, be apolar to all conics $\Phi$, and so must be the vertices of the diagonal triangle of the pencil $\Phi$.

a basic $K^4$ touches S it touches every degree of the pencil on S. We have then,

the four basic $d^4$'s tangent to S touch at the base points of the pencil $\Phi$.

[illegible] on a point. [illegible] by the Veronese [illegible] perspective, taken with

three then sets of projection,
three then regen meet in a point.
Since the lines (on a space $\alpha, \lambda, \beta$) 123
and 456 are polar lines in the
quadric $(\alpha\alpha\lambda)$ it follows that the
points 12 and 45 are on a tangent
plane in every point of them
would denote $\gamma$ in the following
theorem,

The six common tangents of the
six conjugate systems B of A
in a pencil $\Phi$, the base points
of which are the points of contact
of $\gamma$ tangent cones $K^4_\gamma$ to C, the
vertices of the triangle of reference
of $\Phi$ are the points of a quadric
surface S of the Steinerian curve
$(C^{6,11})$

23. Since $K^4_\gamma$ touches $\Sigma$ in

$p$ lie on a two-way $P$ of order $9$. Any space $\alpha$ on $p$ cuts $P$ in a cubic curve and the line $p$ cuts a cubic curve $P'$ in $\alpha$. Any plane $S$ on $p$ and in $\alpha$ cuts $P'$ in eight points, but $P'$ must $p$ in two points, the two points on $p$ in which basic $K^4$'s touch a plane there are in basic $K^7$ that are a line $p$ and $S$ cuts it in the other point not on $p$ therefore,

The locus of points of intersection with a plane $S$ of the $\infty^3$ $K^4$'s becomes a $C^3$ in a cubic surface, $\Sigma$.

$\Sigma$ has a series of straight lines on the locus of its intersections with $S$ of the $\infty^2$ $K^4$'s that are in general three in number, the quantities

Thus a conic $K^4$ on 8 points of $\Delta$. $K^4$ goes through, in general, 14 points of the configuration $\Delta$, namely 8 of the pencils on 123 and 6 on the points where these planes meet $c$, forming a basis for $K^4$ through $\Gamma$. It will contain the 8 points of the base pencil $\Sigma$, and touch at 8 points the tangent to a conic $K^4$ through them.

94. Consider the set of 6 conics $K^4$ on projected from 6 base lines 12, say. They are projected by quadrics through 12 as a nodal line, and project into curves on 3 on 123, 124, 125, 126 points of intersection with 3 of the 6 conics $K^4$ (one of the 6 is one of the generators

[illegible] to [illegible] to the [illegible] O.
[illegible].

Points of intersection with
a plane S of its biscant bas
K's are partners in the Cremona
[illegible] determined by the
Cayley pencil of cones of the
configuration [illegible] or 2. The
six [illegible] are cones of [illegible]
[illegible] a point of [illegible] take [illegible],
[illegible], [illegible], [illegible].

[illegible] that the
[illegible] is a septic curve. Its
[illegible] determined by
[illegible] this septic curve
It carries the line of [illegible] into
a curve of [illegible] the [illegible]
[illegible]-points of [illegible] [illegible]
[illegible] [illegible], the [illegible] Seven

then having the point $p_1$ on a what remarked. Hence,

The points $123, 124, 125, 126$ L, $M, N$, are on a conic.

This is the conic into which the conic $\Phi'$ [illegible] projected from 12 13 [illegible] in this way a configuration of conics with the same set of points and the same incidence conditions.

We also proved in part II that the locus of join of corresponding point on the cente 2 is a line quadric, W say. W touches the six lines of set since $\Phi$ and $\Phi'$, the second generating pencil $\Phi'$ being conics on $123, 124, 125, 126$. Since W is symmetrically related to $\Phi$ & $\Phi'$ it touches also the 10 lines

of $\Delta$ Points 123 and 456 are corresponding points of $\Sigma$, $W$ touches the ten lines $\binom{123}{456}$. It obviously contains the three lines of the diagonal triangle of $\Sigma$. But it is more specially related to these lines, as we shall see.

The configuration $\Delta_1$, dual to $\Delta$ determines a linear range of conics. Here we have a line cubic $\Sigma_1$, and a point cubic $W_1$. The configuration $\Delta_1$ may be considered as the section by a plane of the complete six-point in three dimensions. Conics on the six point meet any plane $\pi$ in a three fold infinite system of conics, that is, in conics upon a range. The

cent quadric on the six points, and hence $123$ and $456$ are apolar to all conics of the range. This range must therefore be the Cremona range of $\Delta_1$. The $R^3$ on the six points carries a double infinity of quadrics, and hence the three points in which it meets a conic, a double infinity of conics apolar to the range — are common to the three pairs. Hence these points are likewise the diagonal points of the range. Now a Weddle pentahedron is known to contain the $10$ on its six nodes, the $15$ lines joining them, and the $10$ lines joining pairs of planes $123$ $456$ on them. The section by a $\Delta$ of the Weddle

face with nodes at the six points must coincide with $W_1$, since it has 24 points in common with it. Further, the bordering surface touches the osculating planes of the $\delta^3$ at its nine nodes, and cuts it in nine conics in harmonic pairs. Hence,

The tangents to the quartic $W_1$ at the three diagonal points of the Cayley tangent cubic from a point, the six points of the plane in each figure to the $K^3$ on the six points determining the configuration $\Delta_1$. $W_1$ meets each of the three diagonal lines in harmonic pairs.

The theorem as to the quadratic relation of the Hessian surfaces $H, H'$ may be stated dually for the quantics $U$ and $S$.

The nine points $H$ determined by cubic $K^{(3)}$ on a plane $\Delta$ touches the three lines of the diagonal triangle of the Cayleyan pencil of conics in three points on a line. The two tangents to $H$ from a point of this diagonal 3-point, besides the two diagonal lines on the point, are apolar to these two lines.

These pencils are [illegible] from the form of the equation given in part 4. The apolarity relation [illegible] that one of the Conics

$$(\alpha x)^2 + \lambda(\beta x)^2 = 0$$

is on the reference three points. Let this be $(\alpha x)^2$. We have then

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = 0$$

and equation (13), part II, becomes,

$$\Sigma \pm \alpha_{23} F_1 F_2 (P_1^2 P_2^2) - 2 \times (\alpha_{12} \beta_{12} - \alpha_{13} (\alpha_{12}) \beta_{12}^2 P_1^2 P_2^2 = 0$$

from which the above theorem is a easy inference.

55. Let us return for a moment to the system of quadrics on six point in space. If $i$ three linear conditions on a quadric to touch a given plane at a given point. Hence one quadric on the six points touches $\pi$ at a given point. The two lines cut out of this quadric are parabolic in the Cremona line involution determined by the Carver range. At a point $a$ of the quartic $K_1$ the quadric tangent to $\alpha$ is a cone with vertex at $a$. The two lines cut out of this cone by $\alpha$ are lines

the sextic $\Sigma_1$. One quadric on the six points contains [illegible] lines of space of the second; this quadric to have a node, the lines lie in a complex. We have then,

Generators of quadric cones on six points in three dimensions are in a sextic complex. The complex curve on any plane is the sextic $\Sigma_1$. This complex contains [illegible] to the $K^3$ on the six point double [illegible].

56. [illegible] of a configuration $\Delta$, determine a conjugate three-fold system of conics to which they are apolar. We have seen above what mean-

Conic by the plane $\pi$ is mapped by this scheme on a Steiner quartic surface $\Omega$. The lines 123, 456, being conjugate lines in the range will go by this mapping scheme into a tangent plane section of $\Omega$, touching at $\binom{12}{456}$. The $\pi$ is in this way ten planes and fifteen points. If we indicate the point of $D_1$ by two figures, we see the incidence conditions of these ten planes and fifteen points may be exhibited in the two diagrams,

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| × | 46 | 62 | 24 | 135 | × | × | × |
| 35 | 12 | 64 | 16 | × | 146 | 162 | 124 |
| 51 | 32 | 34 | 36 | × | 346 | 362 | 324 |
| 13 | 52 | 54 | 56 | × | 546 | 562 | 524 |

135 in the second diagram stands for $\binom{135}{246}$, etc. A point in the left hand diagram is incident

with the planes in the right hand
diagram. Starting in the lower
or bottom one, and working in both,
and similarly for the planes above
them in order, the 2 sets of
the 15 corresponding tangent
and tangent to a 12 nodal quartic
surface. Therefore,

Corresponding to the 15
pairs of $\lambda$, map the planes into
a Steiner quartic surface. The
points of $\lambda$ go into 15 points of
it. The lines $12, 34, \ldots$ go into
10 tangent planes apiece of it.
The points and planes thus obtained
form the configuration of nodes
and tropes of a 12 nodal quartic
surface, 15 points lying by fives
on ten conics.

If $\lambda$ is moved at the begin-

ning of this section that two points in which a line $l^4$ meets the space $S_3$ are pairs of one of the involutions on $(x_1, x_2, x_3, x_4, x_5, x_6)$ and $(x_1, x_2, x_3, x_4, x_5, x_6)$ apolar to $Q_2$. At the end of part III we showed that the four joining pairs of points thus determined lie in a quadratic complex whose discriminant Boundary $K^{(3)}$ are in a quartic complex.

It is obvious that these lines may also be defined as the lines met by four fixed generators of the net of spaces on them in pairs of an involution. This property makes the finding of the equation of the four-dimensional complex easy by means of the theory of invariants

for I. III.

The following is a [illegible] theorem.

The complex curve on any plane S is the quartic IV.

We see from equation (28), part III, that if

$$p_{12} - p_{13} - p_{14} = 0$$

involving

$$\pi_{34} - \pi_{42} - \pi_{23} = 0,$$

or

$$p_{12} - p_{23} - p_{31} = 0,$$

involving

$$\pi_{34} = \pi_{14} - \pi_{24} = 0.$$

The equation of the complex is satisfied. When the complex (21) contains [illegible] lies on [illegible] or on a [illegible]. When a complex is dynamically related to the configuration [illegible]

The system by a space $\sigma_3$ of the complex of lines common to basic $K^4$'s contains singly every line on the points of the configuration $T^3$ in $\sigma_3$, and every line on its points.

The system of $\infty^3$ lying,

The complex of lines common to basic $K^4$'s contains, in general, singly, every line meeting a fixed line and every line lying on a fixed space.

This is the minimum general locus. It has on a three space $\sigma_3$ a pencil $K^4$ of the basic system in that space, and hence a degenerate pencil of basic $K^4$'s. The line meeting 12 vertices of $K^3$ of the basic system is $\infty^3$, the $K^3$ on the point whose at least

3456, and hence has a trisecant basic $R^4$. A similar argument shows that our quartic complex contains every line meeting opposite planes 123-4567 etc.

It is now clear by Theorem 1 that the complex of tangents to the $R^{3,5}$'s contains twice all the points that have pinch doubly. The section of our complex with this cubic complex determined by 12, 14, 36, 46, 56 as base is the congruence of order $6 \cdot 4 - 5 \cdot 2 = 14$, each of class $6 \cdot 7 + 10 \cdot 3 = 72$. This is the $(14, 72)$ congruence of tangents to $R^{3,5}$.

13. Consider any cubic curve in space — not on a Bessel $R^3$ — and the $\infty^1$ chords it has. Each point [illegible] and the [illegible]

three planes of points for the
complete base [illegible] of order 6,
through M, the pole of S in to Q. The
locus of these two extra points of inter-
section is a curve [illegible] with a
triple point at M, since the hy-
perplane section $V^3$ S cuts the three
tangents to $V^4$, and the $V^4$ passes
through M by theorem c'). This
locus is cut by a plane in
a through M in eight points on
the cubic curve, and three
points at M. It obviously is the
locus of points in a plane cut by
1, 2, 3, 4, 5, 6, M hyperplanes in points
of an $R^3$, since a $V^4$ projects
into an $R^3$ from any point on
it. Hence,

the locus of points in a
space of four dimensions from

with a cubic point projects into a cubic point of an $R^3$ in a cubic $\Psi$, of order 11.

The surface $\Psi$ shares a close analogy with the Steiner surface in three dimensions. Let us project it from one of its triple points, $M$ say, into a space 3. Let $\kappa$ be any point $\kappa$ of the surface. If we project $(\gamma_1, \gamma_2, \gamma_3)$ from $\kappa$ on a space $\gamma$ through 11 the points $\gamma_1, \gamma_2, \gamma_3, 4_\gamma, 5_\gamma, 6_\gamma$ are on an $R^3$ with $M$ in $\gamma$, and are projected from $M$ by six lines of a quadric cone, $V$, in $\gamma$. Now the lines $\kappa 1$, $\kappa 2$, etc., projected from $M$ into $\gamma$ into the lines $\kappa_\gamma 1_\gamma$, $\kappa_\gamma 2_\gamma$, &c. Since $\kappa 1$, $\kappa 2$, etc., meet $\gamma$ in points of the $R^3$ by $M$, their images in $\gamma$ are points of $V$, $\kappa_\gamma 1_\gamma$, [illegible] $1_\gamma$ is meant to indicate the point in which [illegible]

By in points of the conic $V_2$. In other words, $1_\beta, 2_\beta$, Etc., project from $K_\beta$ on the plane $S_\beta$, any plane in $\beta$, into six points of a conic. Hence,

The surface $\Psi$ projects from any one of its triple points into a Steiner Surface taken twice. The $H$-ic section of $\Psi$ in any space is thereby $M$ projected into a plane section of this Steiner surface. We have then,

The section by a plane is of a complex cone of the sixth order which may be regarded as a plane section of a Steiner surface. The reasons which we have used in another connection.

The $H$ is curve

the section of the diagram $\Delta'$ by, a space $x$ on $M$, is the section ($\Delta = x$) of the [illegible] surface with centres on $10, 20, \ldots 60, 56, M$, with the complex cone $\Gamma$ of tangent through $M$ to space $\Delta^{45}$. The corresponding intersection is the five lines $M16$ $M26$, $M36$, $M46$, $M56$. There are six such [illegible] surfaces corresponding to the six of the base of $\Gamma$, all containing the curve. Two Families $M, 16, 26, 36, 46, 56$, and $M15, 25, 35, 45, 65$ in Correct in their Curve and $156, 256, 356, 456, M56$.

[illegible] has a triple point at each of the seven defining points $1, 2, 3, 4, 5, 6, 7$ say. It contains the $S^4$ on the seven lines Since this $\Delta'$ passes into an

$R^4$ from any one of its point. It touches the 10 intersecting planes of the $R^3$ since the Middle Surface of $R^4$ which it projects from any one of its seven triple points touches the intersecting planes of the $R^3$ on its six nodes — the curve in $R^4$ which the $R^4$ projects, $\Psi$ contains the 21 lines 12 etc., since, projected from a point of one of these lines, the curve point becomes six, and the $R^3$ cuts through by pairs of six points in an $S_3$. $\Psi$ contains the 35 lines $ijkl$ $ijmn$ $ijpq$ $ijrs$ $ijtu$ $ijvw$ $ijxy$ $ijz\alpha$ $ijz\beta$ $ijz\gamma$ $ijz\delta$ $ijz\varepsilon$ $ijz\zeta$ $ijz\eta$ $ijz\theta$ $ijz\iota$ $ijz\kappa$ $ijz\lambda$ $ijz\mu$ $ijz\nu$ $ijz\xi$ $ijz\omicron$ $ijz\pi$ $ijz\rho$ $ijz\sigma$ $ijz\tau$ $ijz\upsilon$ $ijz\phi$ $ijz\chi$ $ijz\psi$ $ijz\omega$ etc., since 1,2,3 project from a point of one of these lines into these points on a line, and 4,5,6,7 into four points on a plane, and the curve on the four points which the four lines on the plane

line must their plane, taken

V

Basic Norm Curves in a Riemannian

57 The theorems of this section, which
fall in this part of our paper, are
mainly of an enumerative character, but their value in the
treatment of higher cases is
sufficiently obvious.

We shall indicate by $S_m$ a
linear space or flat of $m$ dimensions, and by $C^p_m$ a system of
$\infty^m$ dimensions and of order $p$. Of
the manifolds $C^p_m$ which appear
anywhere in the course of the discussion will frequently be referred
to by the symbol $C^p_m$, indicating
their order and dimension.

60. The curve in $S_n$ is a rth

of $n+1$ points to. It has in $S_n$ the $(n+1)$-fold infinity of points $X^n$, the $\left(n^2+n-\frac{n}{2}\right)$-fold infinity of $S_{n-1}$'s $(S_n^{n-1})$, ... [illegible] of the infinity of points $X^n$ [illegible]

a") Basis $(S_n^1, \ldots, S_{n-1}, S_{n-1}, \alpha_1, \ldots, S_{n-2}^{\alpha_2})$ spoken of as the fundamental $S_{n-1}$, $S_{n-1}^{\alpha}$, $\alpha$.

b") Basis $X^n$ ... $n$ points $S_{n-1}^{\alpha}$ spoken of as the $S_{n-1}^{\alpha}$.

c") Basis $X^n$'s tangent $S_{n-1}$ at point of $S_{n-1}$.

$Q_\alpha$ is the quadric spread in $\alpha$ with regard to which the configuration $\Gamma^n$ cut out of the complete $(n+2)$-point by $\alpha$, is self-polar. Running the base points [illegible]

condition on degree, this number is
that $\frac{(n+1)(n+2)}{2}$ is the degree $\nu_n$ of
the $n$-th $\mathrm{DK}_n$, and so for the various
points we have $\nu_n$ $(n)$ of $\Gamma^n$.

This configuration $\Gamma^n$ contains
$\binom{n+1}{2}$ points, $\binom{n+2}{3}$ lines $\binom{n+3}{n+2}$ $\nu_n$'s.
Every $\nu_n$ contains $\binom{n+2}{n+2}$ $p^{n-1}$, each
every $\nu_p$ is coincident with $\binom{n-1}{p}$
$\nu_{n-1}$'s without $p = n$. [1]

We have from theorem 6"),
e.g. upon,

There is one $6_2$ only, one $6_n$
$n^n$ meeting an $\nu_{n-1}$ $n+1$ times.

We may add to this the following:

The locus $\Gamma^{n-1}$ made of an $\nu_{n-1}$
$n+1$ times is common to the
$\infty^1$ system of $\Gamma^{n-1}_{n+1}$'s on the base and
$\infty^1$ containing the $\nu_{n-1}$.

[1] Cayley, loc. cit.

for the $k^{th}$ such one of these $C_{k,i}$'s in $n+2+m-1$ $2n+1$ points will cut $C$ in $k$ points.

6.2. If such curves as $D_p$ or $D_{p'}$ of the complete $(n+1)$ point opposite of $p_1 p_2 \ldots p_n$ and $p$ will have $2n$ fixed points in common — opposite these $n$ points have no points in common. Hence $D_p$ will degenerate into an $n^{ic}$ in $\Delta_p$ on the $p$'s as fixed points in $\Delta_p$ and the $n$ other points in $\Delta_p$ will be opposite $p$, taken with a similarly determined $R^{n}$ in $\Delta_p$. Further degeneration may occur, but this is the simplest, and by far the most important. The $C_{2n}$ may further degenerate only by the breaking up of $R^{n}$ or $R^{n-1}$. In fact we have the theorem

Every base $S_p$ carries a basis-system of $K^m$'s on the $p+1$ base-points in $S_p$ and the points in which $S_p$ meets its opposite $S_q$. All these $K^m$'s are to be regarded as parts of the generate basis $K^m$'s, the complementary $K^q$'s being the system of $d^q$'s on the similarly determined base in the opposite $S_q$.

63. We can find the order of the $C_{n,2}$ in which each $K^m$ [illegible] $K^m$ must [illegible] again by finding the order of its $C_{n,3}$ of intersection with an $S_{n-2}$ of $\Gamma^n$. A basis $K^m$ cannot meet a base $S_{n-1}$, without degenerating, and hence we need to consider only the degenerate basis $K^m$'s. And we evidently need consider only the basis $K^m$'s degenerating into lines and $K^{m-1}$'s and

into cones and $K^{n-2}$'s. Let us assume that the order of the $C_{n-3}$ of further intersection with an $S_{n-2}$ of $T^n$ is $k^{n-1}$ in $S_{n-1}$ on $m_{n-1}$ $K^{n-1}$'s in the $S_{n-1}$ cutting out the $S_{n-2}$ from $T^n$ determine a $C^{m_{n-1}}_{n-3}$ on $S_{n-2}$. All $S_{n-2}$'s on the $S_{n-2}$ in $T^n$ count double in $C^{m_{n-2}}_{n-2}$ since here cones of the pencil in the plane opposite to a cone $S_{n-2}$ touch it, and either of these cones taken with a sheet $K^n$ on the opposite $S_{n-2}$ on $K^n$ satisfying all conditions. Since there are $\binom{n}{n-1} = n$ $S_{n-2}$'s on $S_{n-2}$ of $T^n$, we have the recurrence formulas,

$$m_n = \Sigma_n + m_{n-1},$$
$$= \Sigma_n + \Sigma_{n-1} + \cdots + \Sigma_2 + m_1,$$

and since $m_1 = 1$ we have

$$m_n = \frac{(n-2)(n-1)}{2}$$

this gives us the theorem,

Given $k^{n-1}$ tangent to $x$ meeting $x$ again in $n\frac{(n+3)(n-2)}{2}$ points.

~~3~~ 3. To next ask for the order of the $C_{n,p}$ on which $k^{n-1}$ having $p$ point contact with $x$ meets $x$ again. Call it $m_p^n$. We can find the order of $C_{n,p}$ by finding the order of the $C_{n,p-1}$ on which it meets an $S_{n-2}$ of $T^n$. The $S_{n,p-1}$ of $T^n$ count $p!$-fold in the section, since they are cut out by base $S_{n-2}$ and $p!$ basic $k^{n-1}$'s of the system on the opposite $S_{p-2}$ having $p$ point contact with $x$. There are $n(n+1)\cdots(n+p-1)$ $S_{n,p-1}$'s on an $S_{n-2}$ of $T^n$. There are on $S_{n-2}$ of $T^n$ there

$$n \cdot p-1 \sim i \sim n+p-2$$

spreads belonging to point ranges, & their dimensions in $S_{n-2}$ are $n-2$

and we need not take them into account. On the $S_{n-1}$ cutting out on $S_{n-r}$ we have a basis system of $\infty^{m_p^{n-1}}$ [illegible] $C_{n-p-1}^{m_p^{n-1}}$ on $S_{n-r}$. Hence we have the recurrence formula

$$m_p^n = \binom{n}{n-p+1} p! + m_p^{n-1}$$

giving

$$\begin{aligned} m_p^n &= p!\left[\binom{n}{n-p+1} + \binom{n-1}{n-p} + \binom{n-2}{n-p-1} + \cdots + \binom{p+1}{p}\right] \\ &= p!\left[\binom{n+1}{p} - p - 1\right] \qquad (1) \end{aligned}$$

Hence a [illegible] $V^n$ [illegible] $\infty^{m_p^n}$ $V^p$ [illegible] with a [illegible] again in a $C_{n-p}^{m_p^n}$ where $m_p^n$ has the value given in (1).

64. We shall say that an $K^n$ is orthic to a $C_{n-1}^2 Q$, when every $C_{n-1}^2$ on the $K^n$ [illegible] $Q$. This is equivalent to saying that there is one, and hence $\infty^1$, $(n-1)$-[illegible]

incident in the $R^n$ and spaces to Q. Use of these $(n+1)$ point conditions can be avoided by choosing a point in $R^n$ and taking cones to this point in which the $R^{n-1}$ meets the polar $\omega_n$ of $\omega$ in Q. If $\omega$ has $\binom{n+2}{2}$ conditions on it in $\Sigma^n_{n-1}$ it contains a pencil of $R^n$'s. These cones then

$$\binom{n+2}{2} - \binom{n+1}{2} = n+1$$

linearly independent $C^n_\omega$'s in a pencil $R^{n-1}$. So in this $\binom{n+1}{2}$ conditions on the system of quadrics to be apolar to Q. But because it is $\binom{n+1}{2}$ conditions on the $R^{n-1}$ to be in a cone in Q.

Basis $R^{n-1}$'s are projected from a base point by a system of $\infty^{n-2}\binom{n+1}{2}$ $R^{n-1}$'s on cones of these cones, one through every point of the base. These $C^n_\omega$'s meet in

points in a basic system of $\Gamma^{n+1}$ are an outline base of $\Gamma^n$. We mean by an "outline base" of $\Gamma^n$ a set of $n+1$ points of $\Gamma^n$ such that every $Q_n'$ on them is apolar to $Q_n$. There are $n+2$ such outline bases in $\Gamma^n$. All points of $\Gamma^n$ whose ... contain one ... form an outline base, as 12, 13, 14 ... every $R^{n-1}$ on an outline base is apolar to $Q_n$. The following theorems are sufficiently evident from what we have said in discussing the three and four dimensional cases;

Sets of points $C_n^{(n)}$ ... $n$ points apolar to $Q_n$ and inscribed in $R^{n-1}$'s of the basic system of any outline base of $\Gamma^n$.

All loci determined in a basic $R^{n-1}$'s are completely determined by $Q_n$ and an outline base of $\Gamma^n$.

65. Choosing as reference points in $S_n$ $n$ points of an algebraic curve of $T^{(n)}$ with the $(n+1)$st point as unit point, a point $K^{n-1}$ in the point $x$ using as given parameters $x_1, \dots, x_n$,

$$y_i = \frac{x_i}{1 - x_1 t} \quad (i = 1, \dots, n) \qquad (1)$$

the point $x$ having the parameter value $t = 0$. For indicate the coordinate $Q_n$ by merely

$$Q = 0.$$

The conditions on the coefficients of $Q$ may easily be worked out. The point $y_{n+1}$ of $x = t\,Q$ is

$$\sum y_i \frac{\partial Q}{\partial x_i} = 0 \qquad (1)$$

(3) Each one of the $\binom{n}{n-1}$ $n+1$ points obtained by taking $n$ from a set $(n)$ in $n$. The $(n+1)$ equation in $t$ giving the parameters of these $n+1$ points may be obtained

[illegible]

$$x_1 x_2 \cdots x_n t^{n-1} - f_n t^{n-2} + \cdots + (-)^{n-2} f_3 t + (-)^{n-1} \cdot 2Q = 0 \quad (4)$$

[illegible]

$$f_{i+1} = \sum x_1 \cdots x_{i-1} \left( \frac{\partial Q}{\partial x_1} + \cdots + \frac{\partial Q}{\partial x_i} \right)$$

[illegible]

$$\Delta = \frac{\partial}{\partial x_i}$$

$\Delta$ is the polar [illegible] of the [illegible] $f_{i+1}$ is a $C_{n-2}^{(i+1)}$ on the $C_{n-1}^{(i+1)}$, the locus of points of [illegible] $x^{n-1}$ [illegible] $(i+1)$ [illegible]

The vanishing of an invariant of (4) of degree $i$ and weight $w$ evidently gives as the locus of the point $x$ a $C_{n-2}^{w+i}$.

$$w = \tfrac{1}{2} i(n-1),$$

we have the theorem,

The locus of points [illegible]

Since that the $n+1$ point $x_0 : x_1 : \dots : x_n$ of the $S_n^{(n)}$ determined by $n+1$ points on the curve $K^{n-1}$ of $S_n$ and on which lies in $I^{n-2}$ a $(n-2)$ point on this $K^{n-1}$ as a simply infinite pencil, then a corresponding invariant $L_{i,j}$ is a $\frac{(n-2)^{n-1}}{n-2}$.

The $n$ points of a $S_n^{(n)}$ correspond to $n$ points of the curve $K^n$, one of projecting $n$ points of the $n$ points to a point on the corresponding $K^{n-1}$ of $S_{n-1}$ of $S_{n-1}$ a projection of the $K^n$, and there $n$ points are corresponding. Hence (4) multiplied by $t$:

$$x_1 \quad x_n L t^n - f_n t^{n-1} + \dots = 0 \qquad (5)$$

represent to written a collineation on the $K^n$ the $n$ points of intersection with a $S_n$ the same $K^n$

$\alpha = \chi$. The vanishing of an invariant of (5) must, then, carry with it the vanishing of the corresponding invariant of the $\chi$ point on the basis $R^n$; the invariant of (5) of degree $\mu$ must be of weight $\mu$ in the $\chi$'s.

[illegible]

$$\chi_{n+i} = i(n+2)$$

[illegible]

[illegible] $R^{(n)}$ [illegible] $\chi_{n-2}$ [illegible]

[illegible] point of $\alpha$ $C_{n-2}^{(n+2)}$

The discriminant of (4) is the $C_{n-2}^{(n+3)(n-2)}$ of further intersection with $\alpha$ of basic $R^n$'s tangent to $\alpha$. It [illegible] $C_{n-3}^{6}$ of point[illegible] [illegible]lation of basic $R^n$'s with $\alpha$ [illegible]

of intersection of $\Delta^{n+1}$ S. Hence,

$\frac{(n+2)(n-1)}{2} = 3$.

It may be proved as in part II that a pair of primary curves corresponding points of the $C_{n-1}$ [illegible] $C_n$.

66. The $C_{n-2}$'s $Q_{\alpha+\lambda\beta}$ on a pencil of $S_n$'s

$$(\alpha x) + \lambda(\beta x) = 0$$

lie on the $C^3_{n-1}$,

$$\begin{vmatrix} (\alpha x^2), & (\beta x^2) \\ (\alpha x) & (\beta x) \end{vmatrix} \equiv \sum \pi_{ij} x_i x_j (x_i - x_j) \quad (4)$$

where $\pi_{ij}$ $(i,j = 0,1,2)$ are the coordinates of the $S_{n-2}$ common to $\alpha$ and $\beta$. Now the cubic $k^3$ on $x$ goes through $x$ with the tangent

$$k^2_{ij} = x_i x_j (x_i - x_j)$$

Hence,

$\pi$ is the $S^{n-1}_{n-1}$, (6).

Consider now $S_{n-2}$ (this meets $\alpha$ in a common $S^{n-3}_{n-2}$) which is determined by the $S_{n-2}$ and $S_{n-3}$ in a $C^1_{n-2}$, each of which has $C^1_{n-2}$ $C_{n-2}$ on the $S_{n-1}$, $\alpha$ common to the two $S_{n-2}$'s factors. Hence,

The locus of points of tangency with $\pi$ of tangents meeting $\alpha$ in $S_{n-2}$ is a $C^1_{n-2}$.

Consider an $S_{n-2}$ and an $S_{n-3}$ which have an $S_{n-4}$ in common. There is an $S_{n-1}$, containing both $S_{n-2}$ and $S_{n-3}$. Tangents to basic $K^n$'s in $\alpha$ are tangent to basic $K^n$'s on a basic in $\alpha$ at points of $S_{n-4}$. Hence tangent to one $K^n$ and meeting

$S_{n-3}$ in a $\alpha_{n-2}$ tangent to $\alpha_{n-1}$ in $\alpha$ which must $S_{n-3}$ and have their points of tangency on $Q_\alpha$. The locus of these points of tangency is the intersection of a $C^3_{n-2}$ with $Q_\alpha$ — a $C^6_{n-3}$. The $C^3_{n-1}$ and the $C^3_{n-2}$ determined in $S_n$ by $S_{n-1}$ and $S_{n-2}$ have $3(n-1)$ intersections in a $C^{n-1}_{n-2}$, one of which is the $C^6_{n-3}$ factor. Hence

The locus of points of tangency with $\infty^{n-3}$ of tangents which meet $S_{n-4}$ is a $C^{3(n-3)}_{n-3}$.

If we now reduce the general case, call the number of the $C_{n-x+1}$ of points of tangency with $\infty^{n-x}$ of tangents meeting an $S_{n-x}$, $m_x$. Then $m_{x+1}$ and an $S_{n-x+1}$ which meet an $S_{n-x}$ in a $C^{m_x}_{n-x}$. Hence is an $S_{n-1}$, $m_x$ times containing both $S_{n-x+1}$ and $S_{n-x}$. Now

an $S_n$, containing $n+1$ points of $S_{n-1}$, one further point of $S_{n-r+1}$ [illegible] — [illegible] points of $S_n$ [illegible] contain a $\Delta$ 'spaces' and is on $n$ points, — enough to fix it. Since tangent to basic $K^n$'s [illegible] meeting $S_{n+1}$ are [illegible] tangent to cones $K^{n-1}$ in a [illegible] meeting $S_{n+1}$ [illegible] meeting these points of contact on $C_{n-1}$. The locus of these points of contact is [illegible] $C^{(n-1)n+1}_{n-1}$ [illegible] $C^{(n-1)n+1}_{n-1}$ [illegible] $C^{n-1}_{n-1}$ [illegible] $C^{(n-1)n+1}_{n-1}$ [illegible] $S_n$ [illegible] and $S_{n-r+1}$ [illegible] in a $C^{(n-1)n+1}_{n-r+1}$ [illegible] of which [illegible] $C^{(n-1)n+1}_{n-r+1}$ [illegible]. Hence [illegible] the recurrence formula

$$m_r = 3m_{r-1} - 2m_{r-2}$$

If $m_1 = 2^{n-1} - 1$ [illegible] $(n-1, r)$ [illegible]

by [illegible]

$$m_r = 3(2^{n-1}-1) - 2(2^{n-2}-1) = 2^n - \ldots$$

But (7) holds for $i$ 3 and $i$ ... we have shown. By induction (7) holds in general. Hence,

The locus of points of tangency with basic $R^n$'s of tangents meeting an $S_{n-r}$ is a $C^{(n-1)r+1}_{r+1}$.

The ... is ... and ...

In particular we have,

The locus of points of tangency with basic $R^n$'s of tangents ... a point $(S_0)$ is a $C^{n-1}_{1}$, a ... of order $(n-1)^2$.

The ... projected from a point $S_0$ by a cone $C^{n-1}_{n-1}$. Hence,

Since tangent to ... $R^n$'s ... on a point lie on a $C^{n-1}_{n-1}$-cone of order $2^n - 2$.

67. It is $n-i-1$ conditions ... a basic $R^n$ in $S_n$ to meet an $S_i$. If the locus of basic $R^n$'s ...

ing on $S_{n-p}$ is $\kappa^{n}_{p-1}$ on $S_{p-1}$ meets this locus in $\kappa^{n}_{p}$ points and there are $\kappa^{n}_{p}$ basic $K^{n}$'s meeting an $S_{n-p}$ and $S_{p-1}$ of them. Hence the locus of basic $K^{n}$'s meeting an $S_{p}$ is met by an $S_{n-p}$ in $\kappa^{n}_{p}$ points and the order of this locus is $\kappa^{n}_{p}$. The locus of basic $K^{n}$'s meeting a $C^{n}_{n-p}$ is met by an $S_{p-1}$ in the $\mu\kappa^{n}_{p}$ points on $S_{p-1}$ by the basic $K^{n}$'s of the locus meeting $S_{p}$ and $C^{n}_{n-p}$. The order of this locus is $\mu\kappa^{n}_{p}$ for since the $C^{n}_{n-p}$ is not specially situated the $\mu\kappa^{n}_{p}$ points are distinct, i.e., there are $\mu\kappa^{n}_{p}$ of the $\kappa^{n}_{p}$. We have seen that basic $K^{n}$'s have $p$ points in common with an $S_{p}$ and meet a $C^{p}_{n-p}$ and meet a again in points of a $C^{m-p}_{n-p}$, $m \geq n$ the meeting points in all are $C^{p}_{n-p}$.

does not meet a curve $C_{n-p}$, since on $V^n$ cannot have $p$-point contact with a if $i < p$.

The locus of basic $k^{n-1}$ meeting $C^p_{n-p}$ is a $C^{n-1}_{n-p+1}$ which meets $\alpha$ in $C^{p,n-1}_{n-p}$ taken $p$ times and in $C^{n-1}_{n-p}$. Hence

$$k^n_p = \frac{p \cdot p! + m^n_p}{p!} - (n_p + 1) - 1. \qquad (8)$$

That is,

Basic $k^{n-1}$ [illegible] on $S_{n-p}$ lie on a $C^{n-1}_{n-p+1}$ which meets $C^p_{n-p}$ $k^n_p$ times [illegible]

There are $k^n_p$ basic $k^{n-1}$ that meet an $S_{n-p}$ [illegible] $S_p$.

Basic $k^{n-1}$ [illegible] $c = \frac{1}{f}$, lie on a $C^{k^n_p}_p$.

The identity [illegible] theorems

$$k^n_p = k^n_{p+1}$$

is evident from the form of (8).

Basic $K^m$'s meeting a $C^p_{n,p}$ lie on a $C^{p+1}_{n,p+1}$ provided the $C^p_{n,p}$ is not specially located in the space. In the latter case a reduction in the order will occur.

The following theorems are easily verified by the methods of part II,

Basic $K^m$'s that lie on a $C^m_{n-1}$ cut it again in a $C^p_{n-1}$ where

$$p = m(m+1)\left(\frac{n-1}{2}\right)\left(\frac{n+1}{3}\right) - m$$

Basic $K^m$'s tangent to a $C^m_{n-1}$ touch at points of a $C^p_{n-1}$ where

$$p = m(m+1)\left(\frac{n-1}{2}\right)\left(\frac{n+1}{3}\right) - m(m+1)(n-1)$$

The multiplicity of a base $\mathfrak{z}_r$ in $C^{p+1}_{n,p+1}$ is $\binom{n-r}{p}$.

68. The argument of § 55, part II, obviously holds in $n$ dimensions. It has only then first to locate a point from which no $n$ points

$S_n$ project into $n+3$ [illegible] of $R^{n-1}$ is a $C_2$ with these properties.

1) It has $(n-1)$-fold point at the $n+3$ points.

2) It projects from one of these points into a $\Psi^{n-1}$ taken twice over.

If $m_n$ is the order of $\Psi^n$ then the recurrence formula

$$m_n = n - 1 + 2m_{n-1}$$

This gives,

$$m_n = n - 1 + 2(n-2) + 4(n-3) + \cdots + 2^n(n+1)$$

$$= \left[1 + 2 + 4 + \cdots + 2^n\right]n$$

$$- \left[1 + 2\cdot 2 + 3\cdot 4 + \cdots + (n-1)2^{n-1}\right]$$

The sum of the series in the first brackets is $2^{n+1} - 1$. The sum of the series in the second brackets may be found as follows, viz.

$$S = 1 + 2\cdot 2 + \cdots + (n-1)2^{n-2}$$

$$= 1 + 2 + 4 + \cdots + 2^{n-2} + 2\left[1 + 2\cdot 2 + 3\cdot 4 + \cdots + (n-2)2^{n-3}\right]$$

$$= 2^{n-1} - 1 + 2\left[2 - (n-1)2^{n-2}\right]$$

whence $2^n = (n-2)2^{n-1} + 1$

and $m_n = n(2^{n-1} - 1) - (n-2)2^{n-1} - 1$

$$= 2^n - n - 1.$$

Thus

The locus of points in $S_n$ from which $n+3$ points project into $n+3$ points of $\ldots S^{n-1}$ is a $C_2^{2^n - n - 1}$.

This locus has properties closely analogous to those of the $\Psi$ of $n-1$ dimensions, as we shall see.

We have also,

Since a bisecant to $C_3^{2^{n-1}-n}$ cuts on a point $a$ in $S_n$ lie on a $C_3^{2^{n-1}-n}$.

The points of intersection with these line of this bisecant locus in $S^{n-1}$ are on the $C_2^{2^n - n - 1}$, $\Psi^n$.

69. By counting of constants and [illegible] of a point

number of cases $K^n$'s on a $C^2_{n-1}$. An $K^n$ in $S^n$ is determined by

$$(n+1)^2 - 4$$

constants. It is $2n+1$ conditions on the $K^n$ to lie on a given $C^2_{n-1}$. It is $n-2$ conditions on a curve lying on a $C_{n-1}$ to pass through a given point of the $C_{n-1}$. Hence

An $K^n$ on a $C^2_{n-1}$ in $S_n$ is determined by

$$\frac{(n+1)^2 - 4 - (2n+1)}{2} \qquad n+2$$

points, a number which is finite in a finite number of positions.

By a theorem of part I, a $K^n$ having $(n-1)$-point contact with a $C^2_{n-1}$, touches at points of a $C^{2n-1}_1$ of the $C^2_{n-1}$. If this $C^{2n-1}_1$ exists, it must be nothing but a $K^{n-1}$, since a $K^n$ having $(n-1)$ point contact with the $C^2_{n-1}$ has

$n+2+n-1 = 2n+1$

points in common with it, [illegible] lie on it. Hence,

There are $(n-1)!$ basic $K^n$'s or- in basis $C^n_{n-1}$.

70. Let $S_{n-1}$ be a basis $K^{n-1}$ meeting the $S_{n-2}$ $i$ times but on a $C^{n-1}_{n-2}$ $k$ times counting that it is a $C^{i,n-1}_{n-2}$. If $i$ is odd a point $P$ may be found on $S_{n-1}$ but not on $S_{n-2}$ by passing from the $C^{i,n-1}_{n-2}$ in which the $S_{n-1}$ meets a basis $S_{n-2}$ to the $C^{i,n}_{n-1}$ in which $S_{n-1}$ meets a basis $S_{n-2}$ in an $S_{n-1}$ hence $K^{n-1}$ on $S_{n-1}$ meeting $S_{n-2}$ $i$ times lies on a $C^{i,n}_{n-1}$. Consider a basis $S_{n-2}$ and the opposite $S_i$. $S_{n-2}$ meets $S_i$ in an $S_{i-1}$. Any basis $S_{n-1}$ meets $S_{i-1}$ $i-1$ times. $S_{n-1}$ meets the space $S_{n-1}$ in an $S_{n-2}$. The various $K^{n-1}$'s on $S_{n-1}$ meeting this $S_{n-2}$ $i$ or $i-2$ times pass through a Hence $S_{n-2}$ must $K^{n-1}$ [illegible] with the

Completion rules, $k$ $r$ times, can be $C^{r,k}_{n-r-1}$. There are $\binom{k}{i+1}$ terms $\Delta_{n-i}$ in a term $S_{n-1}$. This gives the formula

$$T^{k}_{r} = T^{k-1}_{r} + \binom{k}{1} I^{k-1}_{r-1} + \binom{k}{2} I^{k-2}_{r-2} + \cdots + \binom{k}{r} I^{k-r}_{0}, \qquad (7)$$

by which $T^{k}_{r}$ may be calculated in any particular case. The following values of $I$ are in accordance with the form (7),

$$I^{n}_{0} = 1$$
$$I^{n}_{n} = 0 \qquad (n \neq 0)$$
$$I^{0}_{0} = 1$$
$$I^{m}_{n} = 0 \qquad (m < n)$$
$$I^{n}_{n-1} = n$$

the following general formula value for $T^{n+1}_{n-r}$ being obtained by induction from (7):

$$T^{n+1}_{n-r} = \frac{r!(n+1) - (r+1)!\sum_{i=1}^{r}\frac{1}{i} + \sum_{i=1}^{r}(-)^{i-1}\frac{(r!)^2(i-1)!}{(i!)^2(r-i)!}(r-i+1)^{n+1}}{(r!)^2}$$

71. Basis $K^{n-1}$ meeting $r$ $S_{n-1}$'s lie on a $C^{n-1}_{n-r}$; let us assume that it is a $C^{n-1}_{\lambda n-r}$. An argument similar to the above gives us a formula by means of which $\pi^n_r$ may be calculated. Let us examine the $C_{n-1}$ in which one $C_n$ meets a base $S_{n-1}$. The $r$ $S_{n-2}$'s met $S_n$ in $r$ $S_{n-2}$'s and these give a $C^{n-1}_{\pi n-r-1}$, which belongs to our system. Consider a base $S_{n-i}$ and the opposite $S_{i-1}$. $i-1$ $S_{i-2}$'s met $S_i$ in $i-1$ $S_{i-2}$'s. These can be chosen from the $r$ $S_{n-2}$'s in $(i-1)$ ways. There are $\sigma^i_{i-1}/i$ $K^i$'s in $S_i$ meeting these $S_{i-1}$'s. The remaining $r-i+1$ $S_{n-2}$'s meet $S_{n-i}$ in $r-i+1$ $S_{n-i}$'s. Basis $K^{n-i}$'s on $S_{n-i}$ meeting these $S_{n-i}$'s lie on a $C^{(n-i)r-i+1}_{n-i}$. There being $(K^i)$ $S_{n-i}$'s on a base

the formula,

$$\tau_r^n = \sigma_r^{n-1} + \binom{n}{1}\binom{r}{1}\sigma_{r-1}^{n-2}\sigma_1/_2 + \binom{n}{2}\binom{r}{2}\tau_{r-1}^{n-2}\sigma_2/_3$$
$$+ \quad + \binom{n}{r}\binom{r}{r}\sigma_0^{n-r-1}\sigma_{r+1}^{r+1}/_{r+1} \quad (10)$$

the following values of special $\sigma$'s are in accordance with the fact,

$$\sigma_0^n = 1$$
$$\sigma_n^n = 0 \qquad (n \neq 0)$$
$$\sigma_0^0 = 1$$
$$\sigma_n^m = 0 \qquad (n > m)$$
$$\sigma_{n-1}^n = n\,\sigma_{n-2}^n.$$

It is easy to calculate any special $\tau$ or $\sigma$ from (9) or (10), taking with the table of special values. Suppose for example we require the value of $\sigma_3^9$. In the first place,

$$\sigma_3^9 = 4\,\pi_2^9.$$

From (10) we have,

$$\pi_2^9 = \sigma_2^8 + \binom{4}{1}\binom{2}{1}\sigma_1^7\sigma_1^7/_2 + \binom{4}{2}\binom{2}{2}\sigma_0^6\sigma_2^6/_3$$

In

## Biographical Note.

James Ryals Conner was born at Dublin, Georgia, October 15, 1878. He graduated from the University of Georgia with the degree of A.B. in 1898. In October, 1906, he entered the Johns Hopkins University as a candidate for the degree of Doctor of Philosophy, with Mathematics as his principal subject, and with Physical Chemistry and Philosophy as first and second subordinate subjects, respectively. He was held a University fellowship in Mathematics during the years 1907 1909.

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